AN APPROXIMATE VERTEX AMPLITUDE FROM THE SCHWINGER-DYSON EQUATIONS OF QUANTUM ELECTRODYNAMICS

By

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An approximate set of invariant functions for the dressed vertex amplitude was found. An asymptotic solution to the unrenormalized Schwinger-Dyson equations of Quantum Electrodynamics was obtained which joined smoothly with the solutions found by a perturbation technique. The photon propagator is approximated by its form near the mass shell. The vertex equation was separated from higher order members of the hierarchy at the second order in the coupling constant with the aid of H. S. Green's generalization of Ward's Identity. No infinities were substracted to obtain the solutions. The function multiplying the matrix $\underline{\tilde{\gamma}}^{\lambda}$ is found to be dominant everywhere.

CHAPTER I
INTRODUCTION

Before the arrival of quantum mechanics, there were two major problems that seem to point out the existence of flaws in the classical theory of electromagnetic fields. They were the blackbody radiation and the theory of electrons in atoms. The introduction of the finite quantum of action by Planck successfully resolved the first of these problems. The difficulties connected with describing the electron and other entities as "particles" are much more serious. Basically, the nature of the treatment of the electron as a classical particle with finite mass and extent leads to a dilemma. If we assume the electron to be a point without structure, then, in classical theory, the total energy of its electromagnetic field becomes infinite implying an infinitely massive particle. On the other hand, if we assume the electron to have a finite extent, its interaction with the electromagnetic field created by its own charge distribution will produce stresses tending to explode the charge.

The quantum theory of the electron and the electromagnetic field was introduced in the 1920s by Heisenberg and Pauli, Dirac and others. Unfortunately, they were not able to solve the equation that they derived in a manner that was free of infinite quantities. Although the energy of the electromagnetic field diverged as the logarithm of the electron radius in the quantum theory instead of as its reciprocal as it does in the classical theory, a second infinity appeared that is associated with the charge. This infinity appeared in integrals that diverged quadratically as the upper limit increased.[1]

According to Dirac's hole theory,[2] the creation of an electron-positron pair by a photon may be interpreted in the following way. The vacuum consists of an infinite sea of negative-energy electrons. A photon may be absorbed by this vacuum raising a negative energy electron to a positive energy state and therefore creating a hole in the vacuum. The latter will appear as a positron, so that we obtain an electron-positron pair. Thus the vacuum may be considered as a sort of "polarizable" medium, because it potentially contains electron-positron pairs. A photon may now interact with this polarizable vacuum even if its energy is not sufficient to create a real electron-positron pair. In this case only a "virtual" pair is created and this annihilates soon afterwards. Although the apperance of virtual pairs improved the divergence of the self-energy, it introduced new problems that do not have their counterpart in the classical theory. With the introduction of virtual pairs, it was found that the polarizability of the vacuum is infinite.

The Dirac theory also predicts that, for hydrogen-like atoms, states with the same total quantum number n and angular momentum j are degenerate (same energy). It was noted, however, that the polarization of the vacuum discussed in the preceding paragraph would split this degeneracy. In particular the $2^2S_{1/2}$ and the $2^2P_{1/2}$ levels should be separated by a small amount. In 1947, Lamb and Retherford[3] made a direct measurement of this separation. They found that the $2^2S_{1/2}$ is above the $2^2P_{1/2}$ by 1058 megahertz. Actual theoretical calculations using a form of a perturbation expansion of the equations of quantum electrodynamics (QED) of the splitting gave rise to divergent integrals. Bethe[4] circumvented the problem by simply limiting the range of integration over the divergent integrals. Bethe reasoned that at energies larger than the rest energy of the electron, relativistic effects become very important and have to be included. These inclusions, at the end, will amount to the introduction of a cut-off in the integration limits. The final results were, therefore, dependent on a cut-off parameter. This dependency was logarithmic in

the cut-off parameter and thus insensitive to the actual value of the parameter. This parameter could be described as the maximum energy of a photon that was emitted and absorbed by the atomic electron of hydrogen. He set this maximum energy to be the rest energy of the electron. With this technique, Bethe arrive at approximately the value measured by Lamb and Retherford (the so-called "Lamb shift").

The fundamental equations of QED can be written in different forms. One way of doing this is by writing the perturbation expansion in a series of powers of the electron charge. This expansion can be written in terms of propagators (Green's function) as developed by Feynman[5] and interaction sites at which three "particles" can interact. Schwinger[6] and Dyson[7] derived an infinite hierarchy of integral equations that describes the interaction of an electron with the radiation field. In the future, we will refer to this hierarchy as the Schwinger-Dyson hierarchy as it is usually known. The equations in this hierarchy can be solved by straightforward iteration to obtain the perturbation series. This perturbative approach always leads to infinite integrals.

Numerous attempts were made to eliminate the divergencies in a rigorous manner in the period following the invention of quantum field theory around 1925 until after World War II. This problem was solved about the time of the Lamb and Retherford experiments, by the development of the renormalization theory. The idea of renormalization can be interpreted as a rearrangement of the perturbation series so that the new series converges and has finite terms. This rearrangement amounts to expanding the electron and photon propagators and their joining vertex around the mass shell (*i.e.* for values of the photon energy-momentum near zero and electron's square energy-momentum near its experimental square mass). The infinity associated with the charge is removed by rescaling the propagators, wave functions and vertex parts[8,9] with the introduction of the so-called

renormalization constants Z_1, Z_2 and Z_3. They are associated with the vertex function, electron propagator and photon propagator respectively. Attempts to calculate these constants using perturbation theory have led to the conclusion that these constants must be infinite! The renormalization theory emphasized the covariant aspects more strongly. Schwinger, Tomonaga[10] and Feynman,[11] independently developed the first Lorentz covariant schemes designed to eliminate the divergencies in a more acceptable manner (the renormalization theory).

Renormalization theory has enjoyed a remarkable success in the calculation of numerous effects such as the Lamb shift, the anomalous magnetic moment, the hyperfine structure of the hydrogen atom, and other relativistic phenomena. Quantum electrodynamics has become a model for other field theories to follow. It is important then, to study the underlying mathematical structure of QED in order to better understand field theories in general. The success of the non-Abelian gauge theories in unifying the electromagnetic interaction with the weak interaction further encourages the efforts to understand and resolve the ambiguities of QED. The current theories of electro-weak and strong interactions are based on the same underlying mathematical structure.

If the mathematical techniques (perturbation theories) used in QED were complete and satisfactory theory, it would be a mathematically rigorous and logically consistent structure which allows at least in principle the calculation of all radiative processes. Because calculations have been based on the rearrangement of infinite series that may not remove all of the infinite quantities (Z_1, Z_2, and Z_3 for example), this cannot be said without reservations of the theory in the present state of development.

There are two major points of view regarding the infinities found in QED. One states that there is something fundamentally wrong in the foundations of the

theory. The other assumes that all the difficulties arise from the use of inadequate mathematical methods in solving the fundamental equations of the theory. The first point of view states that the solution to the problem must be found in a modified theory or in a completely new one (if possible). The arguments against this radical approach are found in the actual success of renormalization theory in calculating such effects as the Lamb shifts, the anomalous magnetic moment of the electron, and the fine-structure constant. In order to get a flavor of the kind of agreement that can be accomplished with this theory the following results are presented:[12]

1. Lamb shift in hydrogen

$$\delta E_{exp} = 1\ 057\ 845\ (9\ )\, kHz\ , \tag{1.1}$$

$$\delta E_{th} = 1\ 057\ 849\ (11)\, kHz\ , \tag{1.2}$$

2. Electron's anomalous magnetic moment

$$a_{exp} = 1\ 159\ 652\ 200\ (\ 40) \times 10^{-12}\ , \tag{1.3}$$

$$a_{th} = 1\ 159\ 652\ 460\ (127) \times 10^{-12}\ , \tag{1.4}$$

3. Fine structure constant

$$\alpha_{exp}^{-1} = 137.035\ 993\ (5)\ , \tag{1.5}$$

$$\alpha_{th}^{-1} = 137.035\ 989\ (3)\ , \tag{1.6}$$

where the subscripts "exp" and "th" stand for experimental and theoretical values. The quantity enclosed in parentheses represents the uncertainty in the final digit of numerical value. The value of α^{-1} is based on the very accurate measurements of 2e/h (0.03ppm) by the ac Josephson effect. As can be seen from the results given previously, theoretical calculations using renormalization theory match the

experimental results on the order of 0.1ppm. No other theory, to our knowledge, can claim such a success.

If we accept this remarkable agreement between theory and experiment as evidence that the fundamental equations are correct, we are led to favor the conclusion that the infinities that arise in the theory come from an inadequate way of solving the fundamental equations. The study of other model field theories has led to the suspicion that the perturbation expansions after renormalization become series that are only asymptotically convergent. This has led to a search for a better technique of solving the equations. The technique should, in principle, allow the calculation of the bare (noninteracting) mass, charge and constants Z_1, Z_2 and Z_3.

In this search, Gell-Mann and Low[13] sought to demonstrate that the renormalization constants are infinite. They stated that, although they could not rule out the possibility of infinite coupling constants, its was possible to isolate a necessary condition for which the vacuum polarization is finite. In a long series of papers, Johnson, Baker and Willey,[14–17] extended their work and showed that if an eigenvalue condition is satisfied, then all renormalization constants in QED can be finite. The eigenvalue condition defines a function that is the coefficient of the logarithmically divergent integral appearing in the photon propagator calculated in massless QED due to only those graphs with one closed fermion loop. This eigenvalue condition is expressed in terms of the bare coupling constant. Adler[18] then showed that the zero, if it exists, must be an essential singularity of this function with all its derivatives zero at the singularity. The existence of this function has not been proven. Nevertheless, it is possible to speculate that the existence of an infinite order zero will never be seen in any finite order of perturbation expansion. These results generate great interest in a number of people[19] for finding, in a non-perturbative manner, the solutions to the Schwinger-Dyson equations. From

them, Z_1, Z_2, and Z_3 could be identified and it could be determined whether or not infinities are inherent in the theory.

Following this approach, H. S. Green, J. F. Cartier, and A. A. Broyles started the long term project of solving the Schwinger-Dyson hierarchy of equations using a non-perturbative method. The equations in the hierarchy are written in terms of *momentum space* variables, that is, in terms of Fourier transforms. Figure 1.1 shows a general *block diagram* that can be use in describing the state of development of the project. Each block represents an important step in the solution of the hierarchy.

Starting with the Schwinger-Dyson hierarchy of equations (block 1), H.S. Green, J.F. Cartier, and A.A. Broyles[20](herein referred to as Ref. I) were able to determine the unrenormalized electron propagator using an approximate form of the photon propagator. In order to obtain this solution, two approximations were made (block 2). Their first step was to truncate the infinite hierarchy with the aid of Ward's identity.[21] Ward's identity relates the vertex function to the next lower equation in the hierarchy (*i.e.* to the electron propagator) in the limit of zero momentum transfer to the electromagnetic field. The second approximation replaced the photon propagator with its mass-shell form.

With these approximations, they were able to calculate the electron propagator over the entire range of the variables upon which it depends (block 3). They found that in order to have a solution, the bare mass of the electron must be set equal to zero and a particular gauge must be chosen, the so-called Landau gauge. These results are in agreement with the findings of Baker and Johnson.[17] As an additional result of the determination of the electron propagator, the value of the renormalization constants Z_1 and Z_2 were calculated. They conclude that within the boundaries of the approximations made, $Z_1 = Z_2 = 1$. No infinities appeared in obtaining the solutions.

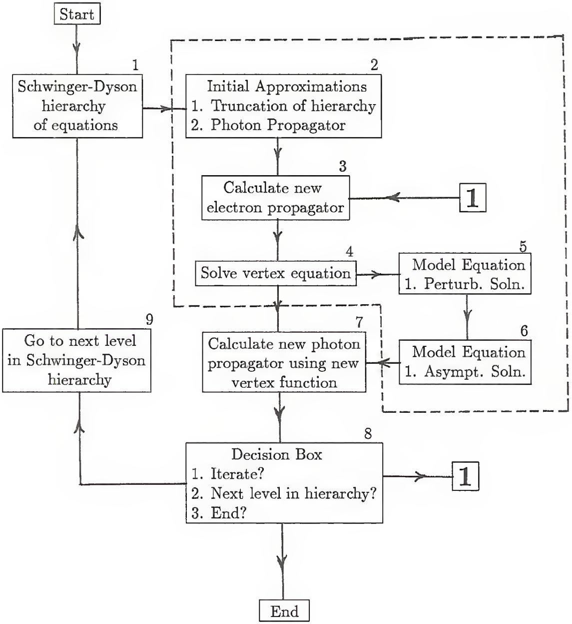

Figure 1.1 Block Diagram of iteration method for solving the Schwinger-Dyson hierarchy of equations.

In order to determine the photon wave-function renormalization constant, associated with the photon propagator Z_3, it is necessary to devise a more accurate form for the photon propagator. The Schwinger-Dyson equation for it involves, however, the electron propagator and vertex amplitude. The electron propagator found in Ref. I is available, but a vertex amplitude must be found. It is possible to demonstrate, by using symmetry and invariance arguments, that this amplitude can be expressed in terms of eight scalar functions of the electron and photon momenta. Some progress has already been made on this subject. Using the same type of approximations which led to the determination of the electron propagator, in Ref. I, and a generalized form of Ward's identity derived by H. S. Green,[22] J. Cartier et al.[23,24] (herein referred to as Ref. II), determined all eight scalar functions needed to define the vertex function in the perturbation region. Since a direct analytical solution to the vertex amplitude is, at the present moment, not feasable (block 4), a somewhat different approach was used. Making some reasonable assumptions about the behavior of the vertex function near the mass shell, an approximate equation to the vertex equation was derived. We call this equation the *model equation* (block 5). The solutions to this equation will indicate the functional form of the vertex amplitude and, in principle, could be used as starting solutions in a *numerical* iteration procedure. Also, a parametrized version of these solutions can be use in a minimization procedure.

The model equation was solved analytically by an iteration procedure and tested to see if the the approximations made were valid. The solutions found proved to represent reasonably well the vertex function near the mass shell and to be consistent with the Ward Identity. Once again, no infinities were incurred in obtaining these solutions.

The next logical step to follow is to find the form of the vertex function in the asymptotic (i.e. large momentum) region and which join smoothly with the

perturbation solutions found in Ref II (block 6). This is the problem that we address in this work. The knowledge of the form of the vertex function in the asymptotic region will open the possibility of determining the photon propagator, the renormalization constant Z_3 and thus the bare charge of the electron (block 7). Finally, to show that the procedure is self-consistent, at least one iteration is necessary. With the new vertex function and photon propagator a recalculation of the electron propagator is possible. The present stage of the project, including the present work, is delimited in the block diagram by the broken line.

The present work is arranged in the following way. Chapter II gives a general description of the method used in truncating the Schwinger-Dyson hierarchy of equations. In this chapter, we also show how to convert the truncated hierarchy into a set of coupled differential equations and the derivation of the model equation. In Chapter III, the model equation is transformed into a set of nine *scalar* differential equations. Also, a simplified version of these differential equations is found to be valid in the asymptotic region. In Chapter IV, the solutions to these equations are found. In the last chapter we summarize and discuss the nature of the solutions.

It is felt that this work can make a positive contribution to a better understanding of the interaction between electrons and photons, and that it will shed some light on the question: Is QED a finite and mathematically sound theory? In addition, the techniques developed may prove to be useful in the solution of other problems in QED.

CHAPTER II

DERIVATION OF THE MODEL EQUATION

2-1 The Schwinger-Dyson Equations

Quantum Electrodynamic Theory (QED) deals with the description of the interaction of light and matter. Such interaction in its most *elementary* stage studies the electron-photon interaction. This description is best handled mathematically using the concept of a propagator for each of the involved "particles" and an *interaction site* called a vertex. The relationships between these propagators (Green's functions) and the vertex are expressed in a hierarchy of equations known as the Schwinger-Dyson equations. To express formally these equations it is necessary to define the meaning of a propagator for each of the "particles" and the vertex function.

The probability of finding an electron at some point in space-time given that it was at a different point in space-time can be computed from its Feynman propagator or Green's function. This propagator satisfies a differential equation analogous to the wave function equation. In coordinate space the wave function of a noninteracting electron $\underline{\Psi}$ satisfies the differential equation

$$(i\nabla\!\!\!/ \; - m_0)\underline{\Psi} \; = \; 0 \tag{2.1}$$

where $\nabla\!\!\!/ \; = \; \underline{\gamma}^\mu \; \partial/\partial x_\mu$ and $\underline{\gamma}^\mu$ are Dirac matrices. The noninteracting photon wave equation is written as

$$\Box^2 \underline{A}^\mu \; = \; 0 \tag{2.2}$$

where $\Box^2$ is the D'Alambertian, defined as

$$\Box^2 \;=\; g^{\mu\nu}\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial x^\nu} \tag{2.3}$$

and

$$g^{\mu\nu} = \begin{cases} +1 & \text{if } \mu \;=\; \nu \;=\; 0; \\ \;\;0 & \text{if } \mu \;\neq\; \nu; \\ -1 & \text{if } \mu \;= \nu \;\neq\; 0\;. \end{cases} \tag{2.4}$$

The electron propagator satisfies an analogous equation

$$(i\nabla\!\!\!/ \;-\; m_0)\; \underline{S}_0(\bar{x}', \bar{x}) \;=\; \delta^4(\bar{x} - \bar{x}')\;. \tag{2.5}$$

Taking the Fourier transform of Eq.(2.5) in momentum space it is found that the noninteracting electron propagator is given by

$$\underline{S}(\bar{p}) = \frac{1}{\not{p} - m_0}\;. \tag{2.6}$$

The noninteracting photon propagator satisfies a similar equation

$$\Box^2 D_0(\bar{x} - \bar{x}') = i\delta^4(\bar{x} - \bar{x}'), \tag{2.7}$$

and the Fourier transform of the photon propagator yields

$$D_0(\bar{q}) = -\frac{1}{\bar{q}^2}. \tag{2.8}$$

These two solutions provide a complete description of the electron and photon when there exists no interaction between them. When interaction is allowed, an infinite hierarchy of non-homogeneous integro-differential coupled equations arises. The exact solutions of these equations were not known. This hierarchy of equations was formulated in their pioneering work by Schwinger[6] and Dyson[7] and was known thereafter as the Schwinger-Dyson hierarchy.

Using the notation of Bjorken and Drell[25,26] these integral equations appear as

$$\underline{S}(\bar{p}) = \underline{S}_0 + \underline{S}(\bar{p})\ \underline{\Sigma}(\bar{p})\ \underline{S}(\bar{p}) \tag{2.9}$$

or equivalently

$$\underline{S}^{-1}(\bar{p}) = \underline{S}_0^{-1}(\bar{p}) - \underline{\Sigma}(\bar{p}) \tag{2.10}$$

where

$$\underline{\Sigma}(\bar{p}) = \frac{ie_0^2}{(2\pi)^4} \int \underline{\Gamma}^{\lambda}(\bar{p},\bar{q})\underline{S}(\bar{q})D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^{\mu}\ d^4q\ , \tag{2.11}$$

$$D_{\mu\nu}(\bar{k}^2) = (D_0)_{\mu\nu}(\bar{k}^2)\ +\ (D_0)_{\mu\alpha}(\bar{k}^2)\ \Pi^{\alpha\beta}\ D_{\beta\nu}(\bar{k}^2)\ , \tag{2.12}$$

$$\Pi^{\mu\nu}(\bar{k}^2) = \frac{ie_0^2}{(2\pi)^4} \int Tr\left[\underline{\gamma}^{\mu}\ \underline{S}(\bar{q})\ \underline{\Gamma}^{\nu}(\bar{q},\bar{q}+\bar{k})\ \underline{S}(\bar{q}+\bar{k})\ \right] d^4q\ , \tag{2.13}$$

$$\underline{\Gamma}^{\mu}(\bar{p},\bar{q}) = \underline{\gamma}_0^{\mu} + \underline{\Lambda}^{\mu}(\bar{p},\bar{q})\ , \tag{2.14}$$

and

$$\begin{aligned}
\underline{\Lambda}^{\mu}(\bar{p},\bar{q})\ &=\ \frac{ie_0^2}{(2\pi)^4} \int D^{\nu\eta}(\bar{k}^2)\underline{\Gamma}_{\nu}(\bar{p},\bar{p}-\bar{k})\underline{S}(\bar{p}-\bar{k}) \\
&\quad \times \underline{\Gamma}^{\mu}(\bar{p}-\bar{k},\bar{q}-\bar{k})\underline{S}(\bar{q}-\bar{k})\underline{\Gamma}_{\eta}(\bar{q}-\bar{k},\bar{q})\ d^4\bar{k} \\
&\quad + \cdots \int \cdots \int \underline{\Gamma}^{\delta}(\bar{p},\bar{p}-\bar{k})\underline{S}(\bar{p}-\bar{k}) \cdots \underline{\gamma}^{\mu} \cdots \\
&\quad \times \underline{\Gamma}_{\eta}(\bar{q}-\bar{k}_n,\bar{k}_{n-1})\underline{S}(\bar{p}-\bar{k}_n)\underline{\Gamma}_{\nu}(\bar{q}-\bar{k},\bar{q}) \\
&\quad \times d^4\bar{k}_n \cdots d^4\bar{k}_1 (2\pi)^{-4n} + \cdots\ .
\end{aligned} \tag{2.15}$$

The zero subscript follows all bare quantities, that is, those functions or constants which are associated with noninteracting particles. An overbar is used to represent four-vectors and an underline to represent matrices. Here $e_0\underline{\Lambda}^{\mu}$ represents the sum of all possible three-external-point, nodeless Feynman diagrams. A node is a bare vertex that if removed, leaves the diagrams separated into at least two unconnected parts. It is possible to relate $\underline{\Gamma}^{\mu}$ to a four-point diagram to either

a dressed electron, or a dressed photon lines. These diagrams can be separated out of the four-point nodeless diagram $e_0\underline{E}^{\lambda\nu}$ leading to the equation

$$\underline{\Gamma}^{\lambda}(\bar{p},\bar{p}+\bar{k}) = \underline{\gamma}_0^{\lambda} + \frac{ie_0^2}{(2\pi)^4}\int D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^{\nu}\underline{S}(\bar{q})$$

$$\times\left\{\underline{\Gamma}^{\lambda}(\bar{q},\bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\underline{\Gamma}^{\nu}(\bar{q}+\bar{k},\bar{p}+\bar{k}) + \underline{E}^{\lambda\nu}(\bar{q},\bar{p}-\bar{q},\bar{p}+\bar{k})\right\}\, d^4q\ . \tag{2.16}$$

The next step in the hierarchy relates the four-point diagram to the five-point and so on. In addition to the Schwinger-Dyson integral equations, there exist some relations between amplitudes whose numbers of external points differ by one. The lowest one relates the vertex amplitude $\underline{\Gamma}^{\mu}$ to the electron propagator $\underline{S}$, and it is attributed to Ward and Takahashi.[21] The one relating the four-point amplitude $\underline{E}^{\lambda\nu}$ and the three-point amplitude $\underline{\Gamma}^{\lambda}$ was derived by H.S. Green.[22] Ward's identity can be expresses as follows:

$$(p_\mu - q_\mu)\underline{\Gamma}^{\mu}(\bar{p},\bar{q}) = \underline{S}^{-1}(\bar{p}) - \underline{S}^{-1}(\bar{q}) \tag{2.17}$$

or in terms of $\underline{\Lambda}^{\mu}(\bar{p},\bar{q})$

$$(p_\mu - q_\mu)\underline{\Lambda}^{\mu}(\bar{p},\bar{q}) = \underline{\Sigma}(\bar{q}) - \underline{\Sigma}(\bar{p})\ . \tag{2.18}$$

An analogous equation can be written for the four-point amplitude in terms of the vertex amplitude and is given by

$$\begin{aligned}(p_\lambda + k_\lambda - q_\lambda)\underline{E}^{\lambda\nu}(\bar{p},\bar{k},\bar{q}) &= \underline{\Gamma}^{\nu}(\bar{q}-\bar{k},\bar{q}) - \underline{\Gamma}^{\nu}(\bar{p},\bar{p}+\bar{k})\\ &= \underline{\Lambda}^{\nu}(\bar{q}-\bar{k},\bar{q}) - \underline{\Lambda}^{\nu}(\bar{p},\bar{p}+\bar{k})\end{aligned} \tag{2.19}$$

and

$$k_\nu\ \underline{E}^{\lambda\nu}(\bar{p},\bar{k},\bar{q}) = \underline{\Gamma}^{\lambda}(\bar{p}-\bar{k},\bar{q}) - \underline{\Gamma}^{\lambda}(\bar{p},\bar{p}+\bar{k})\ . \tag{2.20}$$

Similar relationships exist for the remaining n-point diagrams. These identities exactly define the longitudinal components of the n-point diagram in terms of the

(n-1)-point diagrams. If we orient the μ^{th} axis along the n^{th} direction of $\bar{k}$, then it can be shown that $\underline{E}^{\lambda\nu}$ can be written as

$$\begin{aligned}\underline{E}^{\lambda\nu}(\bar{p},\bar{k},\bar{q}) = (p_\mu + k_\mu - q_\mu)\underline{E}^{\mu\nu}(\bar{p},\bar{k},\bar{q})(p^\lambda + k^\lambda - q^\lambda) \\ + k_\mu\ \underline{E}^{\lambda\mu}(\bar{p},\bar{k},\bar{q})k^\nu + \underline{E}_t^{\lambda\nu}(\bar{p},\bar{k},\bar{q})\end{aligned} \tag{2.21}$$

where $\underline{E}_t^{\lambda\nu}$ is transverse to both of the attached photon four-momenta. Substituting equations (2.19) and (2.20) into (2.21) gives

$$\begin{aligned}\underline{E}^{\lambda\nu}(\bar{p},\bar{k},\bar{q}) = &\left[\underline{\Gamma}^\lambda(\bar{p}-\bar{k},\bar{q}) - \underline{\Gamma}^\lambda(\bar{p},\bar{p}+\bar{k})\right](p^\nu + k^\nu - q^\nu) \\ &+\left[\underline{\Gamma}^\lambda(\bar{p}-\bar{k},\bar{q}) - \underline{\Gamma}^\lambda(\bar{p},\bar{q}+\bar{k})\right]\ k^\nu \\ &+\underline{E}_t^{\lambda\nu}(\bar{p},\bar{k},\bar{q})\end{aligned} \tag{2.22}$$

where

$$k_\nu \underline{E}_t^{\lambda\nu}(\bar{p},\bar{k},\bar{q}) = 0\ . \tag{2.23}$$

Equation (2.20) can also be written in derivative form if we orient the coordinate axis so that the ν^{th} axis lies along $\bar{k}$. Then dividing by k^ν and taking the limit as k^ν vanishes gives

$$\underline{E}^{\lambda\nu}(\bar{p},0,\bar{q}) = -\frac{\partial\underline{\Gamma}^\lambda(\bar{p},\bar{q})}{\partial p_\nu} - \frac{\partial\underline{\Gamma}^\lambda(\bar{p},\bar{q})}{\partial q_\nu}\ . \tag{2.24}$$

Similarly, letting k_λ approach $(q_\lambda - p_\lambda)$ in equation (2.19) gives

$$\underline{E}^{\lambda\nu}(\bar{p},\bar{q}-\bar{p},\bar{q}) = -\frac{\partial\underline{\Gamma}^\lambda(\bar{p},\bar{q})}{\partial p_\nu} - \frac{\partial\underline{\Gamma}^\lambda(\bar{p},\bar{q})}{\partial q_\nu}\ . \tag{2.25}$$

Using similar arguments, it is possible to separate $\underline{\Gamma}^\mu$ into longitudinal and transverse parts so that

$$\underline{\Gamma}^\mu(\bar{p},\bar{q}) = \left[\underline{S}^{-1}(\bar{p}) - S^{-1}(\bar{q})\right]\ k^\mu + \underline{\Gamma}_t^\mu(\bar{p},\bar{q}) \tag{2.26}$$

where $k_\mu \underline{\Gamma}_t^\mu = 0$ or explicitly

$$\underline{\Gamma}_\ell^\mu(\bar{q}+\bar{k},\bar{p}+\bar{k}) = \left[\underline{S}^{-1}(\bar{q}+\bar{k}) - \underline{S}^{-1}(\bar{p}+\bar{k})\right]\ \frac{(q^\mu - p^\mu)}{(\bar{q}-\bar{p})^2}\ . \tag{2.27}$$

Using equations (2.16), (2.21) and (2.27) leads to the following equation for the vertex amplitude:

$$
\begin{aligned}
\underline{\Gamma}^{\lambda}(\bar{p},\bar{p}+\bar{k}) = \underline{\gamma}_0^{\lambda} + \frac{ie_0^2}{(2\pi)^4}\int D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^{\mu}\underline{S}(\bar{q})\Big\{&\underline{\Gamma}^{\lambda}(\bar{q},\bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\\
&\times\left[\underline{\Gamma}_t^{\nu}(\bar{q}+\bar{k},\bar{p}+\bar{k}) + \left[\underline{S}^{-1}(\bar{p})-\underline{S}^{-1}(\bar{q})\right]\ \frac{(q^{\nu}-p^{\nu})}{(\bar{q}-\bar{p})^2}\right]\\
&-\left[\underline{\Gamma}^{\nu}(\bar{q}+\bar{k},\bar{p}+\bar{k})-\underline{\Gamma}^{\nu}(\bar{q},\bar{p})\right]\ k^{\lambda}\\
&+\left[\underline{\Gamma}^{\lambda}(2\bar{q}-\bar{p},\bar{p}+\bar{k})-\underline{\Gamma}^{\lambda}(\bar{q},2\bar{p}+\bar{k}-\bar{q})\right]\ (p^{\nu}-q^{\nu})\\
&+\underline{E}_t^{\lambda\nu}(\bar{q},\bar{p}-\bar{q},\bar{p}+\bar{k})\Big\}\ d^4q\ .
\end{aligned}
\tag{2.28}
$$

2-2 Approximations to the Schwinger-Dyson Equations

Although equation (2.28) is exact, it involves another equation in the hierarchy through the four-point amplitude $\underline{E}_t^{\lambda\nu}$. It is possible to close the hierarchy of equations by noticing in equation (2.28) that the integrand has a maximum at the pole $(\bar{p}-\bar{q})=0$ and therefore it is reasonable to approximate $\underline{E}_t^{\lambda\nu}$ by

$$
\underline{E}_t^{\lambda\nu}(\bar{q},\bar{p}-\bar{q},\bar{p}+\bar{q}) \sim \underline{E}_t^{\lambda\nu}(\bar{q},0,\bar{p}+\bar{q})\ . \tag{2.29}
$$

Since λ is transverse to $\bar{k}$, and ν is transverse to $(\bar{p}-\bar{q})$, $\underline{E}_t^{\lambda\nu}$ can be written as

$$
\underline{E}_t^{\lambda\nu} = \underline{E}^{\lambda\nu} - \frac{k^{\lambda}k_{\eta}}{\bar{k}^2}\underline{E}^{\eta\nu} - \frac{(p_{\eta}-q_{\eta})(p^{\nu}-q^{\nu})}{(\bar{p}-\bar{q})^2}\underline{E}^{\lambda\eta}\ . \tag{2.30}
$$

Substituting equations (2.24) and (2.29) into (2.30) gives

$$
\underline{E}_t^{\lambda\nu}(\bar{q},\bar{p}-\bar{q},\bar{p}+\bar{q}) \sim -\left[g^{\lambda}_{.\eta}g^{\nu}_{.\alpha} - \frac{k^{\lambda}k^{\eta}}{\bar{k}^2}g^{\nu}_{.\alpha} - g^{\lambda}_{.\eta}\frac{(p_{\alpha}-q_{\alpha})(p^{\nu}-q^{\nu})}{(\bar{p}-\bar{q})^2}\right]\frac{\partial\underline{\Gamma}^{\eta}(\bar{q},\bar{q}+\bar{k})}{\partial q_{\alpha}} \tag{2.31}
$$

which can be used to eliminate $\underline{E}_t^{\lambda\nu}$ in equation (2.28). This equation coupled with equations (2.9) to (2.16) could be solved in principle for $\underline{\Gamma}^{\lambda}$,$\underline{S}$ and $D_{\mu\nu}$.

It is possible at this stage to solve equation (2.28) for the component of $\underline{\Gamma}^\lambda(\bar{p}, \bar{p}+\bar{k})$ transverse to k^λ and then use equation (2.26) to construct $\underline{\Gamma}^\lambda$. Taking the transverse components of $\underline{\Gamma}^\lambda$ will eliminate all terms proportional to k^λ. Substituting equation (2.31) into (2.28) gives

$$\begin{aligned}\underline{\Gamma}_t^\lambda(\bar{p}, \bar{p}+\bar{k}) = \underline{\gamma}_{0t}^\lambda + \frac{ie_0^2}{(2\pi)^4}\int D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^\mu \underline{S}(\bar{q})\Big\{ &\underline{\Gamma}_t^\lambda(\bar{q}, \bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k}) \\ &\times\left[\underline{\Gamma}_t^\nu(\bar{q}+\bar{k}, \bar{p}+\bar{k}) + \left[\underline{S}^{-1}(\bar{p}) - \underline{S}^{-1}(\bar{q})\right]\frac{(q^\nu - p^\nu)}{(\bar{q}-\bar{p})^2}\right] \\ &+\left[\underline{\Gamma}_t^\lambda(2\bar{q}-\bar{p}, \bar{p}+\bar{k}) - \underline{\Gamma}_t^\lambda(\bar{q}, 2\bar{p}+\bar{k}-\bar{q})\right](p^\nu - q^\nu) \\ &+\frac{\partial\underline{\Gamma}_t^\lambda(\bar{q}, \bar{q}+\bar{k})}{\partial q_\alpha}(p_\alpha - q_\alpha)(p^\nu - q^\nu) - \frac{\partial\underline{\Gamma}_t^\lambda(\bar{q}, \bar{q}+\bar{k})}{\partial q_\nu}\Big\}d^4q\ .\end{aligned} \tag{2.32}$$

The final goal would be to solve this equation with Eqs.(2.9) to (2.13) self-consistently. To this end, it will be advantageous to solve a much simpler equation obtained by making some seemingly sound approximation. The solutions to this equations will shed light on the form and behavior of the complete solution in both the perturbation and asymptotic limit.

The last step in the procedure of reducing the Schwinger-Dyson equations to a more tractable form was to convert the integral equations into a set of differential equations with appropriate boundary conditions. This method was first developed by H. S. Green in connection with the Beth-Salpeter equation. It was first used in the study of the Schwinger-Dyson equations by Bose and Biswas.[27]

2-3 The Model Equation

It is possible to recast equation (2.16) using Ward and Green's identities for both terms in the square brackets. Replacing $\bar{p}$ by $\bar{q}$ and making use of Ward's identity gives

$$\underline{\Gamma}^\lambda(\bar{q}, \bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\underline{\Gamma}_\nu(\bar{q}+\bar{k}, \bar{p}+\bar{k}) \sim \underline{\Gamma}^\lambda(\bar{q}, \bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\frac{\partial\underline{S}^{-1}(\bar{q}+\bar{k})}{\partial q^\nu}\ . \tag{2.33}$$

Substituting this equation and the expression for $\underline{E}^{\lambda\mu}(\bar{q}, 0, \bar{q}+\bar{k})$ in equation (2.16) and taking the transverse component yields

$$\underline{\Gamma}_t^{\lambda}(\bar{p}, \bar{p}+\bar{k}) = \underline{\gamma}_{0t}^{\lambda} + \frac{ie_0^2}{(2\pi)^4} \int D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^{\mu}\underline{F}^{\lambda\nu}(\bar{q}, \bar{q}+\bar{k})d^4q \tag{2.34}$$

where

$$\underline{F}^{\lambda\nu}(\bar{q}, \bar{q}+\bar{k}) = \underline{S}(\bar{q})\frac{\partial}{\partial q^{\nu}}\left\{\underline{\Gamma}_t^{\lambda}(\bar{q}, \bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\right\}\underline{S}^{-1}(\bar{q}+\bar{k}) \ . \tag{2.35}$$

Using the renormalization group,[26] arguments have been presented to show that the asymptotic form of $D_{\mu\nu}$ is the same as that near the mass shell. One of these arguments is that the contribution of the vacuum polarization to the Lamb Shift is only 27 megahertz out of 1058. The vacuum polarization gives a measurement of the departure of the photon propagator from its mass shell form. It is also observed, once again, that the integrand of equation (2.34) is largest when the argument of $D_{\mu\nu}$ vanishes. Hence, the photon propagator is approximated by

$$D_{\mu\nu}(\bar{q}) \cong Z_3\left[-g_{\mu\nu} + \frac{q_{\mu}q_{\nu}}{\bar{q}^2}\right]\frac{1}{\bar{q}^2} \ , \tag{2.36}$$

where Z_3 is the photon renormalization constant.

The gauge is chosen to be that found necessary to obtain a finite solution to the electron propagator equation[20] with a vanishing bare mass for the electron. In the above reference the electron propagator is found to be given by

$$\underline{S}^{-1}(\bar{p}) \cong -A(\bar{p}^2) + B(\bar{p}^2)\not{p} \tag{2.37}$$

where

$$A(\bar{p}^2) = m\left|\frac{\bar{p}^2}{m^2} - 1\right|^{3\alpha(m^2-\bar{p}^2)/4\pi\bar{p}^2} \ , \tag{2.38}$$

$$B(\bar{p}^2) \cong 1 \ , \tag{2.39}$$

and $\alpha \approx 1/137$ is the fine structure constant. With these approximations it is possible to transform equation (2.32) to a differential equation by taking the D'Alambertian in momentum space (see Eq.(2.3)) on both sides of Eq.(2.34). Using the identity

$$\frac{\partial}{\partial(p_\mu - q_\mu)}\frac{\partial}{\partial(p_\nu - q_\nu)}\ln(\bar{p}-\bar{q})^2 = 2\frac{g_{\mu\nu}}{(\bar{p}-\bar{q})^2} - 4\frac{(p_\mu - q_\mu)(p_\nu - q_\nu)}{(\bar{p}-\bar{q})^4} \tag{2.40}$$

it is possible to write Eq.(2.36) as

$$D_{\mu\nu}(\bar{p}-\bar{q}) = -\frac{1}{2}Z_3\frac{g_{\mu\nu}}{(\bar{p}-\bar{q})^2} - \frac{1}{4}Z_3\frac{\partial}{\partial(p_\mu - q_\mu)}\frac{\partial}{\partial(p_\nu - q_\nu)}\ln(\bar{p}-\bar{q})^2 \tag{2.41}$$

Application of the D'Alambertian to Eq.(2.41) gives

$$\Box^2 D_{\mu\nu}(\bar{p}-\bar{q}) = -\frac{1}{2}Z_3 g_{\mu\nu}\Box^2\frac{1}{(\bar{p}-\bar{q})^2} - \frac{1}{4}Z_3\partial_\mu\partial_\nu\Box^2\ln(\bar{p}-\bar{q})^2 \ . \tag{2.42}$$

Substituting the identities

$$\Box^2\frac{1}{(\bar{p}-\bar{q})^2} = i(2\pi)^2\delta^4(\bar{p}-\bar{q}), \tag{2.43}$$

$$\Box^2\ln(\bar{p}-\bar{q})^2 = \frac{4}{(\bar{p}-\bar{q})^2} \tag{2.44}$$

into Eq.(2.42) gives

$$\begin{aligned}\Box^2 D_{\mu\nu}(\bar{p}-\bar{q}) = &- 2\pi^2\ i\ g_{\mu\nu}\delta^4(\bar{p}-\bar{q})Z_3 \\ &- Z_3\frac{\partial}{\partial(p^\mu - q^\mu)}\frac{\partial}{\partial(p^\nu - q^\nu)}\frac{1}{(\bar{p}-\bar{q})^2}\ .\end{aligned} \tag{2.45}$$

Therefore,

$$\Box^2\underline{\Gamma}_t^\lambda(\bar{p},\bar{p}+\bar{k}) = \frac{ie^2}{(8\pi^2)}\left\{-2\pi^2 i Z_3\underline{\gamma}_\nu\underline{S}(\bar{p})\frac{\partial}{\partial p_\mu}\left[\underline{\Gamma}_t^\lambda(\bar{p},\bar{p}+\bar{q})\underline{S}(\bar{p}+\bar{q})\right]\underline{S}^{-1}(\bar{p}+\bar{q})\right\}$$

$$\begin{aligned}&- \frac{ie_0^2 Z_3}{(2\pi)^2}\int\left\{\underline{\gamma}^\mu\frac{\partial}{\partial(p^\mu - q^\mu)}\frac{\partial}{\partial(p^\nu - q^\nu)}\frac{1}{(\bar{p}-\bar{q})^2}\right. \\ &\left.\times\ \underline{S}(\bar{q})\frac{\partial}{\partial k_\nu}\left[\underline{\Gamma}_t^\lambda(\bar{q},\bar{q}+\bar{k})\underline{S}(\bar{q}+\bar{k})\right]\underline{S}^{-1}(\bar{q}+\bar{k})\right\}d^4q\ .\end{aligned} \tag{2.46}$$

In terms of the tensor $F^{\lambda\nu}$, Eq.(2.43) becomes

$$\Box^2\underline{\Gamma}_t^\lambda(\bar{p},\bar{p}+\bar{k}) = \frac{1}{2}\epsilon\underline{\gamma}_\nu\underline{F}^{\lambda\nu}(\bar{p},\bar{p}+\bar{k}) - \frac{ie^2}{(2\pi)^4}\int\frac{\partial}{\partial(p_\nu-q_\nu)}\not{\nabla}\frac{1}{(\bar{p}-\bar{q})^2}\underline{F}^{\lambda\nu}(\bar{p},\bar{p}+\bar{k})d^4q$$
$$= \epsilon\left[\frac{1}{2}\underline{\gamma}_\nu\underline{F}^{\lambda\nu} + \underline{G}^\lambda\right] \tag{2.47}$$

where

$$\underline{G}^\lambda = -\frac{i}{(2\pi)^2}\frac{\partial}{\partial p^\nu}\int\underline{\gamma}^\alpha\frac{\partial}{\partial(p^\alpha-q^\alpha)}\left[\frac{1}{(\bar{p}-\bar{q})^2}\right]\underline{F}^{\lambda\nu}d^4q\ . \tag{2.48}$$

Applying $\not{\nabla}$ to $\underline{G}^\lambda$ and using Eq.(2.43) we obtain

$$\not{\nabla}\underline{G}^\lambda = \frac{\partial}{\partial p^\nu}\underline{F}^{\lambda\nu} \tag{2.49}$$

and hence

$$\not{\nabla}\Box^2\underline{\Gamma}^\lambda(\bar{p},\bar{p}+\bar{k}) = \epsilon\left[\frac{1}{2}\not{\nabla}\underline{\gamma}_\nu\underline{F}^{\lambda\nu}(\bar{p},\bar{p}+\bar{k}) + \frac{\partial}{\partial p^\nu}\underline{F}^{\lambda\nu}(\bar{p},\bar{p}+\bar{k})\right]\ . \tag{2.50}$$

We have define $e^2 = e_0^2 Z_3$ and $\epsilon = e^2/(2\pi)^2 = (\alpha/\pi)$ where α again is the fine-structure constant. It is possible to write Eq.(2.50) in a more symmetrical form as

$$\Box^2\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) = \epsilon\left[\frac{1}{2}\underline{\gamma}_\nu\underline{F}^{\lambda\nu}(\bar{p}_1,\bar{p}_2) + \frac{\partial}{\partial p^\nu}\not{\nabla}^{-1}\underline{F}^{\lambda\nu}(\bar{p}_1,\bar{p}_2)\right] \tag{2.51}$$

where

$$\underline{F}^{\lambda\nu}(\bar{p}_1,\bar{p}_2) = -\underline{S}(\bar{p}_1)\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2)\underline{S}(\bar{p}_2)\underline{\gamma}^\nu + \underline{A}^{\nu\lambda}(\bar{p}_1,\bar{p}_2) \tag{2.52}$$

and

$$\underline{A}^{\nu\lambda}(\bar{p}_1,\bar{p}_2) = \underline{S}(\bar{p}_1)\left\{\frac{\partial\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2)}{\partial p_\nu} - \underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2)\underline{S}(\bar{p}_2)\left[\frac{\partial\underline{S}^{-1}(\bar{p}_2)}{\partial p_\nu} - \underline{\gamma}^\lambda\right]\right\}\ . \tag{2.53}$$

Here $\bar{p}_1$ and $\bar{p}_2$ refer to the outgoing and incoming electron momenta respectively. The tensor $\underline{F}^{\lambda\nu}$ has been written in such a way that it supports further approximations. To first order in ϵ it is well known that near $\bar{p}_1^2 \cong \bar{p}_2^2 \cong m^2$ (i.e.

near the mass shell) $\underline{\Gamma}^{\lambda} \cong \underline{\gamma}^{\lambda} Z_2^{-1}$ and therefore $\frac{\partial \underline{S}^{-1}(\bar{p}_2)}{\partial p^{\nu}} \cong \underline{\gamma}^{\lambda} Z_2^{-1}$. If we assume that this form is correct we can neglect the derivatives of the vertex function; *i.e.* $\frac{\partial \underline{\Gamma}^{\lambda}}{\partial p^{\nu}} \cong 0$. Notice also that to this order of approximation $Z_1 \approx Z_2 \approx 1$.[20] Under these approximations it is reasonable to assume that $\underline{A}^{\lambda\nu} \cong 0$. This leads us to a simpler, more compact equation which we called the model equation for $\underline{\Gamma}^{\lambda}$ and is explicitly written as

$$\Box^2 \underline{\Gamma}^{\lambda}(\bar{p}_1, \bar{p}_2) = -\epsilon \left[\frac{1}{2} \underline{\gamma}_{\nu} \; \underline{\Delta}^{\lambda}(\bar{p}_1, \bar{p}_2) \; \underline{\gamma}^{\nu} + \not{\nabla}^{-1} \underline{\Delta}^{\lambda} \overleftarrow{\not{\nabla}} \right] \tag{2.54}$$

where

$$\underline{\Delta}^{\lambda}(\bar{p}_1, \bar{p}_2) = \underline{S}(\bar{p}_1) \underline{\Gamma}^{\lambda}(\bar{p}_1, \bar{p}_2) \underline{S}(\bar{p}_2) \; . \tag{2.55}$$

We are interested in the solutions to this equation coupled with equations (2.36) and (2.37) in the asymptotic region. These solutions must join smoothly with solutions near the mass shell (*i.e.* perturbation limit). To solve these equations, it is convenient to write them in term of invariant functions. This can be done as follows: let us define the matrices (see Appendix A)

$$\underline{\gamma}_{\mu\nu} = [\underline{\gamma}_{\mu}, \underline{\gamma}_{\nu}] \qquad \underline{\gamma}_{\lambda\mu\nu} = \tfrac{1}{2} \left\{ \underline{\gamma}_{\lambda}, \underline{\gamma}_{\mu\nu} \right\} = i \varepsilon_{\lambda\mu\nu\rho} \underline{\gamma}^5 \underline{\gamma}^{\rho} \tag{2.56}$$

$$\underline{\gamma}_5 = -i \underline{\gamma}_0 \underline{\gamma}_1 \underline{\gamma}_2 \underline{\gamma}_3 \tag{2.57}$$

where $\varepsilon_{0123} = -\varepsilon^{0123} = 1$. Here the symbols $[,]$ and $\{,\}$ represent the commutator and anti-commutator of Dirac's matrices. One can also define the following vectors and tensors

$$C^{\lambda} = \tfrac{1}{4} \, Tr \left[\underline{\Gamma}^{\lambda} \right] \; , \tag{2.58}$$

$$C^{\lambda}_{\mu} = \tfrac{1}{4} \, Tr \left[\underline{\Gamma}^{\lambda} \underline{\gamma}_{\mu} \right] \; , \tag{2.59}$$

$$C^{\lambda}_{\mu\nu} = \tfrac{1}{4} \, Tr \left[\underline{\Gamma}^{\lambda} \underline{\gamma}_{\mu\nu} \right] \; , \tag{2.60}$$

$$C^{\lambda}_{\mu\nu\rho} = \tfrac{1}{4} \, Tr \left[\underline{\Gamma}^{\lambda} \underline{\gamma}_{\mu\nu\rho} \right] \; , \tag{2.61}$$

$$D^\lambda = \tfrac{1}{4}\, Tr\left[\underline{\Delta}^\lambda\right] \ , \tag{2.62}$$

$$D^\lambda_\mu = \tfrac{1}{4}\, Tr\left[\underline{\Delta}^\lambda \underline{\gamma}_\mu\right] \ , \tag{2.63}$$

$$D^\lambda_{\mu\nu} = \tfrac{1}{4}\, Tr\left[\underline{\Delta}^\lambda \underline{\gamma}_\mu{}^\nu\right] \ , \tag{2.64}$$

$$D^\lambda_{\mu\nu\rho} = \tfrac{1}{4}\, Tr\left[\underline{\Delta}^\lambda \underline{\gamma}_{\mu\nu\rho}\right] = i\varepsilon_{\mu\nu\rho\sigma} Tr\left[\underline{\Delta}^\lambda \underline{\gamma}_5 \underline{\gamma}^\sigma\right] \ . \tag{2.65}$$

The tensor equations derived from Eq.(2.54) are

$$\Box^2 C^\lambda = -3\epsilon D^\lambda \ , \tag{2.66}$$

$$\Box^2 C^\lambda_\mu = 2\epsilon D^\lambda_\mu - 2\epsilon\partial_\mu \Box^{-2} \partial^\nu D^\lambda_\nu \ , \tag{2.67}$$

$$\Box^2 C^\lambda_{\mu\nu} = -\epsilon D^\lambda_{\mu\nu} - 2\epsilon\partial_\mu \Box^{-2}\partial^\rho D^\lambda_{\nu\rho} + 2\epsilon\partial_\nu \Box^{-2}\partial^\rho D^\lambda_{\mu\rho} \ , \tag{2.68}$$

$$\begin{aligned} \Box^2 C^\lambda_{\mu\nu\rho} = \quad & -2\epsilon\partial_\mu \Box^{-2}\partial^\sigma D^\lambda_{\nu\rho\sigma} \\ & - 2\epsilon\partial_\nu \Box^{-2}\partial^\sigma D^\lambda_{\rho\mu\sigma} \\ & - 2\epsilon\partial_\rho \Box^{-2}\partial^\sigma D^\lambda_{\mu\nu\sigma} \ . \end{aligned} \tag{2.69}$$

If we write

$$\underline{S}(\bar{p}_1) = \frac{\not{p}_1 + A_1}{(\bar{p}_1^2 - A_1^2)} \ , \qquad \underline{S}(\bar{p}_2) = \frac{\not{p}_2 + A_2}{(\bar{p}_2^2 - A_2^2)} \tag{2.70}$$

where $A_1 = A_1(\bar{p}_1^2)$, $A_2 = A_2(\bar{p}_1^2)$, and also define

$$D = (\bar{p}_1^2 - A_1^2)(\bar{p}_2^2 - A_2^2) \ , \tag{2.71}$$

then equations (2.58) to (2.61) become

$$\begin{aligned} D^\lambda &= \tfrac{1}{4} Tr\left[\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2)(\not{p}_2 + A_2)(\not{p}_1 + A_1)\right]/D \\ &= \quad \left[(A_1 A_2 + \bar{p}_1 \cdot \bar{p}_2) C^\lambda + (A_1 p_2^\mu + A_2 p_1^\mu) C^\lambda_\mu \right. \\ & \qquad \left. - (p_1^\mu p_2^\nu) C^\lambda_{\mu\nu}\right]/D \ , \end{aligned} \tag{2.72}$$

$$D^\lambda_\mu = \tfrac{1}{4} Tr\left[\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2)(\not{p}_2 + A_2)\underline{\gamma}_\mu(\not{p}_1 + A_1)\right]/D$$

$$
\begin{aligned}
&= \quad \big[(A_2 p_{1\mu} + A_1 p_{2\mu})C^\lambda + (A_1 A_2 - \bar{p}_1 \cdot \bar{p}_2) C^\lambda_\mu \\
&\qquad + (p_{1\mu} p_2^\nu + p_{2\mu} p_1^\nu) C^\lambda_\nu + (A_2 p_1^\nu + A_1 p_2^\nu) C^\lambda_{\mu\nu} \\
&\qquad - (p_1^\nu p_2^\rho) C^\lambda_{\mu\nu\rho}\big]/D \ , \qquad (2.73)
\end{aligned}
$$

$$
\begin{aligned}
D^\lambda_{\mu\nu} &= \tfrac{1}{4} Tr\big[\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2)(\not{p}_2 + A_2)\underline{\gamma}_{\mu\nu}(\not{p}_1 + A_1)\big]/D \\
&= \quad \big[-(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})C^\lambda + (A_2 p_{1\nu} - A_1 p_{2\nu}) C^\lambda_\mu \\
&\qquad - (A_2 p_{1\mu} - A_1 p_{2\mu}) C^\lambda_\nu + (A_1 A_2 + \bar{p}_1 \cdot \bar{p}_2) C^\lambda_{\mu\nu} \\
&\qquad - (p_{1\nu} p_2^\rho + p_{2\nu} p_1^\rho) C^\lambda_{\mu\rho} + (p_{1\mu} p_2^\rho + p_{2\mu} p_1^\rho) C^\lambda_{\nu\rho} \\
&\qquad + (A_2 p_1^\rho + A_1 p_2^\rho) C^\lambda_{\mu\nu\rho}\big]/D \ , \qquad (2.74)
\end{aligned}
$$

$$
\begin{aligned}
D^\lambda_{\mu\nu\rho} &= \tfrac{1}{4} Tr\big[\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2)(\not{p}_2 + A_2)\underline{\gamma}_{\mu\nu\rho}(\not{p}_1 + A_1)\big]/D \\
&= \quad \big[(p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu}) C^\lambda_\rho + (p_{1\nu} p_{2\rho} - p_{2\nu} p_{1\rho}) C^\lambda_\mu \\
&\qquad + (p_{1\rho} p_{2\mu} - p_{2\rho} p_{1\mu}) C^\lambda_\nu + (A_2 p_{1\mu} + A_1 p_{2\mu}) C^\lambda_{\nu\rho} \\
&\qquad + (A_2 p_{1\nu} + A_1 p_{2\nu}) C^\lambda_{\rho\mu} + (A_2 p_{1\rho} + A_1 p_{2\rho}) C^\lambda_{\mu\nu} \\
&\qquad + (A_1 A_2 + \bar{p}_1 \cdot \bar{p}_2) C^\lambda_{\mu\nu\rho} + (p_{1\mu} p_2^\sigma + p_{2\mu} p_1^\sigma) C^\lambda_{\nu\rho\sigma} \\
&\qquad + (p_{1\nu} p_2^\sigma + p_{2\nu} p_1^\sigma) C^\lambda_{\rho\mu\sigma} + (p_{1\rho} p_2^\sigma + p_{2\rho} p_1^\sigma) C^\lambda_{\mu\nu\sigma}\big]/D \ . \qquad (2.75)
\end{aligned}
$$

In deriving the above expressions, several matrix identities were used (see Appendix A). It is possible to express these tensors in terms of scalar functions using the following reasoning. To construct the vector function C^λ there is only one vector at our disposal, $\tilde{p}^\lambda$ since the vector k^λ does not appear in the transverse part of $\tilde{\underline{\Gamma}}^\lambda$. Therefore, the most general form of C^λ is

$$
C^\lambda = u \tilde{p}^\lambda \ . \qquad (2.76)
$$

To construct the tensor C^λ_μ we have at our disposal only the tensors, $\tilde{\delta}^\lambda_\mu$, $p_{1\mu}\tilde{p}^\lambda$, and $p_{2\mu}\tilde{p}^\lambda$. Hence, the most general form for C^λ_μ is

$$
C^\lambda_\mu = v \ \tilde{\delta}^\lambda_\mu + (u_2 p_{1\mu} + u_1 p_{2\mu}) \tilde{p}^\lambda \ . \qquad (2.77)
$$

In constructing the general form of $C^{\lambda}_{\mu\nu}$, we notice from Eq.(2.60) and the antisymmetry property of $\underline{\gamma}_{\mu\nu}$ under the exchange of the indices that $C^{\lambda}_{\mu\nu} = -C^{\lambda}_{\nu\mu}$. The only tensors at out disposal to construct $C^{\lambda}_{\mu\nu}$ are: $p_{1\mu}\tilde{\delta}^{\lambda}_{\nu}$, $p_{1\nu}\tilde{\delta}^{\lambda}_{\mu}$, $p_{2\mu}\tilde{\delta}^{\lambda}_{\nu}$, $p_{2\nu}\tilde{\delta}^{\lambda}_{\mu}$ and $p_{1\mu}p_{2\nu}\tilde{p}^{\lambda}$. Thus, the most general form of $C^{\lambda}_{\mu\nu}$ which satisfies the antisymmetry relation is

$$\begin{aligned} C^{\lambda}_{\mu\nu} &= (v_2 p_{1\nu} + v_1 p_{2\nu})\tilde{\delta}^{\lambda}_{\mu} - (v_2 p_{1\mu} + v_1 p_{2\mu})\tilde{\delta}^{\lambda}_{\nu} \\ &\quad + u_3 (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{p}^{\lambda} \ . \end{aligned} \tag{2.78}$$

Similar arguments can be used to conclude that

$$\begin{aligned} C^{\lambda}_{\mu\nu\rho} &= v_3 \big[(p_{1\nu}p_{2\rho} - p_{2\nu}p_{1\rho})\tilde{\delta}^{\lambda}_{\mu} + (p_{1\rho}p_{2\mu} - p_{2\rho}p_{1\mu})\tilde{\delta}^{\lambda}_{\nu} \\ &\quad + (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{\delta}^{\lambda}_{\rho} \big] \ . \end{aligned} \tag{2.79}$$

With these definitions the vertex amplitude takes on the form

$$\begin{aligned} \underline{\tilde{\Gamma}}^{\lambda}(\bar{p}_1, \bar{p}_2) &= \underline{\tilde{\gamma}}^{\lambda}\, v(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + \tilde{p}^{\lambda} u(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + \not{p}_1 \tilde{p}^{\lambda} u_1(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + \not{p}_2 \tilde{p}^{\lambda} u_2(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + \tilde{p}^{\lambda}[\not{p}_1, \not{p}_2] u_3(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + [\not{p}_1, \underline{\tilde{\gamma}}^{\lambda}] v_1(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + [\not{p}_2, \underline{\tilde{\gamma}}^{\lambda}] v_2(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \\ &\quad + i\varepsilon^{\lambda\mu\nu\rho}\underline{\gamma}_{\mu} p_{1\nu} p_{2\rho} v_3(\bar{p}_1^2, \bar{p}_2^2, \bar{p}_1 \cdot \bar{p}_2) \ . \end{aligned} \tag{2.80}$$

A tilde is used over variables to specify transverse parts to $\bar{k}$. Explicitly, the transverse part of p^{λ} is given by $\tilde{p}^{\lambda} = p^{\lambda} - \frac{\bar{p}\cdot\bar{k}}{\bar{k}^2} k^{\lambda}$. Also the tensors D^{λ}, D^{λ}_{μ}, $D^{\lambda}_{\mu\nu}$ and

$D^{\lambda}_{\mu\nu\rho}$ can be written in terms of scalar functions as follows:

$$D^{\lambda} = R\tilde{p}^{\lambda} \ , \tag{2.81}$$

$$D^{\lambda}_{\mu} = s\ \tilde{\delta}^{\lambda}_{\mu} + (R_2 p_{1\mu} + R_1 p_{2\mu})\tilde{p}^{\lambda} \ , \tag{2.82}$$

$$\begin{aligned} D^{\lambda}_{\mu\nu} = {} & (p_{2\nu}\tilde{\delta}^{\lambda}_{\mu} - p_{2\mu})\tilde{\delta}^{\lambda}_{\nu})S_1 + (p_{1\nu}\tilde{\delta}^{\lambda}_{\mu} - p_{1\mu})\tilde{\delta}^{\lambda}_{\nu})S_2 \\ & + R_3(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{p}^{\lambda} \ , \end{aligned} \tag{2.83}$$

$$\begin{aligned} D^{\lambda}_{\mu\nu\rho} = {} & s_3\big[(p_{1\nu}p_{2\rho} - p_{2\nu}p_{1\rho})\tilde{\delta}^{\lambda}_{\mu} + (p_{1\rho}p_{2\mu} - p_{2\rho}p_{1\mu})\tilde{\delta}^{\lambda}_{\nu} \\ & + (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{\delta}^{\lambda}_{\rho}\big] \ . \end{aligned} \tag{2.84}$$

Substituting these expressions into equations (2.72) to (2.75) gives the following results for the scalar functions R,R_1,... *etc.*:

$$\begin{aligned} R = \Big\{ & (A_1A_2 + \bar{p}_1\cdot\bar{p}_2)u + (A_1p_2 + A_2p_1)\cdot(u_1\bar{p}_2 + u_2\bar{p}_1) \\ & + (A_1 + A_2)v + (\bar{p}_1^2 - \bar{p}_1\cdot\bar{p}_2)v_2 - (\bar{p}_2^2 - \bar{p}_1\cdot\bar{p}_2)v_1 \\ & - [\bar{p}_1^2\bar{p}_2^2 - (\bar{p}_1\cdot\bar{p}_2)^2]u_3\Big\}/D \ , \end{aligned} \tag{2.85}$$

$$\begin{aligned} R_1 = \Big\{ & A_1u + A_1A_2u_1 + \bar{p}_1^2u_2 + v + (A_1 - A_2)v_1 - (A_2\bar{p}_1^2 + A_1\bar{p}_1\cdot\bar{p}_2)u_3 \\ & - (\bar{p}_1^2 - \bar{p}_1\cdot\bar{p}_2)v_3\Big\}/D \ , \end{aligned} \tag{2.86}$$

$$\begin{aligned} R_2 = \Big\{ & A_2u + \bar{p}_2^2u_1 + A_1A_2u_2 + v + (A_1 - A_2)v_2 - (A_1\bar{p}_2^2 + A_2\bar{p}_1\cdot\bar{p}_2)u_3 \\ & + (\bar{p}_2^2 - \bar{p}_1\cdot\bar{p}_2)v_3\Big\}/D \ , \end{aligned} \tag{2.87}$$

$$\begin{aligned} R_3 = \{ & -u - A_1u_2 - A_2u_1 + (A_1A_2 + \bar{p}_1\cdot\bar{p}_2)u_3 \\ & + (v_2 - v_1) + (A_1 + A_2)v_3\Big\}/D \ , \end{aligned} \tag{2.88}$$

$$\begin{aligned} S = \Big\{ & (A_1A_2 + \bar{p}_1\cdot\bar{p}_2)v + (A_2\bar{p}_1^2 + A_1\bar{p}_1\cdot\bar{p}_2)v_2 - A_1\bar{p}_2^2 + A_2\bar{p}_1\cdot\bar{p}_2)v_1 \\ & + (\bar{p}_1^2\bar{p}_2^2 - (\bar{p}_1\cdot\bar{p}_2)^2\,)u_3\Big\}/D \ , \end{aligned} \tag{2.89}$$

$$S_1 = \Big\{-A_1v + A_1A_2v_1 - \bar{p}_1^2v_2 - (A_2\bar{p}_1^2 + A_1\bar{p}_1\cdot\bar{p}_2)v_3\Big\}/D \ , \tag{2.90}$$

$$S_2 = \left\{ \underline{A}_2 v + A_1 A_2 v_2 - \bar{p}_2^2 v_1 + (A_1 \bar{p}_2^2 + A_2 \bar{p}_1 \cdot \bar{p}_2) v_3 \right\} / D \ , \tag{2.91}$$

$$S_3 = \left\{ v - A_2 v_1 + A_1 v_2 + (A_1 A_2 + \bar{p}_1 \cdot \bar{p}_2) v_3 \right\} / D \ . \tag{2.92}$$

CHAPTER III

DERIVATION OF THE SCALAR DIFFERENTIAL EQUATIONS

3-1 Definition of Variables

To obtain the scalar differential equations resulting from the model equation, let us define the following variables which will prove to be very useful in describing the asymptotic region. Let

$$x = \sqrt{\bar{p}_1^2 \bar{p}_2^2} \; ; \tag{3.1}$$

$$z = \sqrt{\frac{\bar{p}_1^2}{\bar{p}_2^2}}; \qquad y = \frac{1}{2}(z + z^{-1}) \; ; \tag{3.2}$$

$$z = y + \sqrt{y^2 - 1}; \qquad z^{-1} = y - \sqrt{y^2 - 1} \; . \tag{3.3}$$

The inverse transformations are

$$\bar{p}_1^2 = \bar{p}^2 + 2\bar{p} \cdot \bar{k} + \bar{k}^2 = xz; \qquad \bar{p}_2^2 = \bar{p}^2 - 2\bar{p} \cdot \bar{k} + \bar{k}^2 = xz^{-1} \; , \tag{3.4}$$

so that

$$\bar{p}^2 = xy - \bar{k}^2; \quad \bar{p} \cdot \bar{k} = \tfrac{1}{2}x\sqrt{y^2 - 1}; \quad \bar{p}_1 \cdot \bar{p}_2 = \bar{p}^2 - \bar{k}^2 = xy - 2\bar{k}^2 \; . \tag{3.5}$$

If both $\bar{p}_1^2$ *and* $\bar{p}_2^2$ are time-like or space-like 4-vectors, the variables x and y are real. With these restrictions, the variables y or z are always greater than one. Using this choice of variables, the momentum space is divided into three regions:

$$\begin{cases} \text{perturbation region, } x \approx m^2 \; , \; y \approx 1, \; \bar{k}^2 \approx 0 \; ; \\ \text{inner asymptotic region, } x/\bar{k}^2 \gg m^2 \; , \; y \approx 1 \; ; \\ \text{outer asymptotic region, } x/\bar{k}^2 \gg m^2 \; , \; y \gg 1 \; . \end{cases} \tag{3.6}$$

In terms of these variables the differential operator $\partial_\mu \equiv \partial/\partial p^\mu$ can be written as

$$\partial_\mu = L_\mu \frac{\partial}{\partial x} + M_\mu \frac{\partial}{\partial y} \ , \tag{3.7}$$

where

$$L_\mu = 2\left\{ y \ p_\mu - \sqrt{y^2-1} \ k_\mu \right\} \ , \tag{3.8}$$

$$M_\mu = -2\frac{\sqrt{y^2-1}}{x} \left\{ \sqrt{y^2-1} \ p_\mu - y \ k_\mu \right\} \ . \tag{3.9}$$

If we have a *vector* function W^λ of the form $W^\lambda = w(x,y)\tilde{p}^\lambda$, then

$$\partial_\mu W^\lambda = (\partial_\mu w)\tilde{p}^\lambda + w\tilde{\delta}^\lambda_\mu \tag{3.10}$$

and

$$\Box^2 W^\lambda = \tilde{p}^\lambda \Box^2 w + 2\tilde{\partial}^\lambda w \ , \tag{3.11}$$

where $\Box^2 = \partial^\mu \partial_\mu$ is the D'Alambertian operator. To specify the derivatives of $w(x,y)$ with respect to the x and y variables we use the notation

$$\begin{aligned} w_x &= \frac{\partial w}{\partial x}, \quad w_{xx} = \frac{\partial^2 w}{\partial x^2}, \\ w_y &= \frac{\partial w}{\partial y}, \quad w_{yy} = \frac{\partial^2 w}{\partial y^2}, \\ w_{xy} &= \frac{\partial^2 w}{\partial x \partial y} \ . \end{aligned} \tag{3.12}$$

Using these symbols, it is found that the D'Alambertian operator acting on a scalar function of the variables x and y gives

$$\begin{aligned} \Box^2 w(x,y) = {} & 4(xy - \bar{k}^2)w_{xx} + 4(2y - \frac{\bar{k}^2}{x})w_x + \frac{4\bar{k}^2(y^2-1)}{x^2} w_{yy} \\ & + 4\left[y \ \frac{\bar{k}^2}{x^2} - \frac{(y^2-1)}{x} \right] w_y - 4(y^2-1)w_{xy} \ . \end{aligned} \tag{3.13}$$

With the help of Eq.(3.10) it is easy to see that

$$\begin{aligned} \Box^2 W^\lambda(x,y) = {} & 4(xy - \bar{k}^2)w_{xx} + 4(3y - \frac{\bar{k}^2}{x})w_x + \frac{4\bar{k}^2(y^2-1)}{x^2} w_{yy} \\ & + 4\left[y \ \frac{\bar{k}^2}{x^2} - \frac{2(y^2-1)}{x} \right] w_y - 4(y^2-1)w_{xy} \end{aligned} \tag{3.14}$$

when acting upon a *vector* function, since from Eqs.(3.7), (3.8) and (3.9) gives

$$\tilde{\partial}^\lambda w = 2\left[y\ w_x - \frac{(y^2-1)}{x} w_y\right]\tilde{p}^\lambda\ . \tag{3.15}$$

To distinguish between the two differential operators, we used the subscript s or v to indicate the operator used in Eq.(3.13) or (3.14), respectively. Explicitly,

$$\begin{aligned}\Box_s^2 = {} & 4(xy-\bar{k}^2)\frac{\partial^2}{\partial x^2} + 4(2y - \frac{\bar{k}^2}{x})\frac{\partial}{\partial x} \\ & + \frac{4\bar{k}^2(y^2-1)}{x^2}\frac{\partial^2}{\partial y^2} + 4\left[y\ \frac{\bar{k}^2}{x^2} - \frac{2(y^2-1)}{x}\right]\frac{\partial}{\partial y} \\ & - 4(y^2-1)\frac{\partial^2}{\partial x \partial y}\ ,\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}\Box_v^2 = {} & 4(xy-\bar{k}^2)\frac{\partial^2}{\partial x^2} + 4(3y - \frac{\bar{k}^2}{x})\frac{\partial}{\partial x} \\ & + \frac{4\bar{k}^2(y^2-1)}{x^2}\frac{\partial^2}{\partial y^2} + 4\left[y\ \frac{\bar{k}^2}{x^2} - \frac{(y^2-1)}{x}\right]\frac{\partial}{\partial y} \\ & - 4(y^2-1)\frac{\partial^2}{\partial x \partial y}\ .\end{aligned} \tag{3.17}$$

To recast the nine coupled differential equations that arise from the model equation in terms of the new variables, let us define the operators $\hat{h}$ and $\hat{l}$ by

$$\hat{h} = 2\left\{y\ \frac{\partial}{\partial x} - \frac{(y^2-1)}{x}\frac{\partial}{\partial y}\right\} \tag{3.18}$$

$$\hat{l} = -2\sqrt{y^2-1}\left\{\frac{\partial}{\partial x} - \frac{y}{x}\frac{\partial}{\partial y}\right\} \tag{3.19}$$

and also

$$\hat{f}_\pm = \frac{\partial}{\partial x} \pm \frac{\sqrt{y^2-1}}{x}\frac{\partial}{\partial y}\ . \tag{3.20}$$

Notice that with this notation

$$\tilde{\partial}^\lambda = \tilde{p}^\lambda \hat{h}\ . \tag{3.21}$$

It is easy to show that in terms of these differential operators

$$\partial_\mu = p_{1\mu} z^{-1} \hat{f}_+ + p_{2\mu} z \hat{f}_- = p_\mu \hat{h} + k_\mu \hat{l} \tag{3.22}$$

since

$$\begin{aligned} z^{-1}\hat{f}_+ + z\hat{f}_- &= \hat{h} \\ z^{-1}\hat{f}_+ - z\hat{f}_- &= \hat{l} \ . \end{aligned} \tag{3.23}$$

3-2 Derivation of Differential Equations

At this moment we turn to the problem of transforming Eqs.(2.66) to (2.69) into scalar differential equations. It is easy to see that Eq. (2.66) can be written in terms of the scalar functions u and R, as

$$\Box_v^2 u = -3\epsilon R \ . \tag{3.24}$$

To obtain the scalar differential equations from Eq.(2.67) it is observed that this equation is equivalent to two coupled differential equations. To this end we define the vector function $E^\lambda = \epsilon \tilde{p}^\lambda$. With this definition, equation (2.67) becomes

$$\Box^2 C_\mu^\lambda = 2\epsilon \left[D_\mu^\lambda - \partial_\mu E^\lambda \right] \tag{3.25}$$

coupled with

$$\Box^2 E^\lambda = \partial^\nu D_\nu^\lambda \ . \tag{3.26}$$

It is important at this moment to realize that from Eq.(2.67)

$$\partial^\mu C_\mu^\lambda = 0 \ . \tag{3.27}$$

This follows from a direct application of ∂^μ to C_μ^λ. The proof follows:

$$\begin{aligned} \partial^\mu \Box^2 C_\mu^\lambda = \Box^2 \partial^\mu C_\mu^\lambda &= 2\epsilon \left[\partial^\mu D_\mu^\lambda - \partial^\mu \partial_\mu \Box^{-2} \partial^\nu D_\nu^\lambda \right] \\ &= 2\epsilon \left[\partial^\mu D_\mu^\lambda - \partial^\nu D_\nu^\lambda \right] = 0 \ . \end{aligned}$$

Assuming that $\partial^\mu C_\mu^\lambda$ is finite at $\bar{p}^2 = 0$, and $\bar{p}^2 = \infty$, then the only choice possible is the result stated in Eq.(3.27). In addition to this argument, we find that the tensor C_μ^λ is defined in terms of the scalar function v. In the asymptotic region, the largest contribution to C_μ^λ comes from this function. It is found (see appendix D) that in the asymptotic region the function v approaches a constant, thus supporting the hypothesis that $\partial^\mu C_\mu^\lambda \sim 0$ to leading order in the variable x.

If we apply in general, ∂^α to C_μ^λ we obtain

$$\begin{aligned}\partial^\alpha C_\mu^\lambda &= \tilde{\delta}_\mu^\lambda \partial^\alpha v + \tilde{\delta}^{\lambda\alpha}(u_2 p_{1\mu} + u_1 p_{2\mu}) + (u_1 + u_2)\tilde{p}^\lambda \delta_\mu^\alpha \\ &+ \left[p_{1\mu}\partial^\alpha u_2 + p_{2\mu}\partial^\alpha u_1\right]\tilde{p}^\lambda \ .\end{aligned} \tag{3.28}$$

Evaluating Eq.(3.28) at $\alpha = \mu$ gives

$$\partial^\mu C_\mu^\lambda = \partial^\lambda v + 5(u_1 + u_2)\tilde{p}^\lambda + \tilde{p}^\lambda\left[\bar{p}_1 \cdot \Box u_2 + \bar{p}_2 \cdot \Box u_1\right] = 0 \tag{3.29}$$

or using Eqs.(3.21)

$$\hat{h}(v) + 5(u_1 + u_2) + \bar{p}_1 \cdot \Box u_2 + \bar{p}_2 \cdot \Box u_1 = 0 \ . \tag{3.30}$$

Applying ∂_α to Eq.(3.28) gives

$$\begin{aligned}\Box^2 C_\mu^\lambda &= \tilde{\delta}_\mu^\lambda\left[\Box_s^2 v + 2(u_1 + u_2)\right] + 2\left[p_{1\mu}\partial^\lambda u_2 + p_{2\mu}\partial^\lambda u_1\right] \\ &+ \left[p_{1\mu}\Box_s^2 u_2 + p_{2\mu}\Box_s^2 u_1\right]\tilde{p}^\lambda + 2(\partial_\mu u_1 + \partial_\mu u_2)\tilde{p}^\lambda \ .\end{aligned} \tag{3.31}$$

Using Eq.(3.21) we obtain

$$\begin{aligned}\Box^2 C_\mu^\lambda &= \tilde{\delta}_\mu^\lambda\left[\Box_s^2 v + 2(u_1 + u_2)\right] \\ &+ p_{1\mu}\tilde{p}^\lambda\left[\Box_s^2 u_2 + 2\hat{h}(u_2) + 2z^{-1}\hat{f}_+(u_1 + u_2)\right] \\ &+ p_{2\mu}\tilde{p}^\lambda\left[\Box_s^2 u_1 + 2\hat{h}(u_1) + \quad 2z\hat{f}_-(u_1 + u_2)\right] \ .\end{aligned} \tag{3.32}$$

$$\Box^2 C^\lambda_\mu = \tilde{\delta}^\lambda_\mu \left[\Box^2_v v - 2\hat{h}(v) + 2(u_1 + u_2)\right]$$
$$+ p_{1\mu}\tilde{p}^\lambda \left[\Box^2_v u_2 + 2z^{-1}\hat{f}_+(u_1 + u_2)\right]$$
$$+ p_{2\mu}\tilde{p}^\lambda \left[\Box^2_v u_1 + \quad 2z\hat{f}_-(u_1 + u_2)\right] \; . \tag{3.33}$$

From Eq.(2.82), it follows that the right-hand side of Eq.(3.25) becomes

$$2\epsilon\left[D^\lambda_\mu - \partial_\mu E^\lambda\right] = 2\epsilon\Big\{(S - e)\tilde{\delta}^\lambda_\mu + p_{1\mu}\tilde{p}^\lambda\left[R_2 - z^{-1}\hat{f}_+(e)\right]$$
$$+ p_{2\mu}\tilde{p}^\lambda\left[R_1 - \quad z\hat{f}_-(e)\right]\Big\} \; . \tag{3.34}$$

Also, from a direct application of ∂^ν to D^λ_ν we obtain that

$$\partial^\nu D^\lambda_\nu = \left\{5(R_1 + R_2) + \hat{h}(S) + \bar{p}_1 \cdot \Box R_1 + \bar{p}_2 \cdot \Box R_2\right\} \; . \tag{3.35}$$

Comparing Eqs.(3.33) and (3.34), it becomes evident that we can extract four differential equations for the functions v, u_1, u_2 and e. They are

$$\Box^2_v v - 2\hat{h}(v) + 2(u_1 + u_2) = 2\epsilon(S - e) \; , \tag{3.36}$$

$$\Box^2_v u_1 + \quad 2z\hat{f}_-(u_1 + u_2) = 2\epsilon\left[R_1 - \quad z\hat{f}_-(e)\right] \; , \tag{3.37}$$

$$\Box^2_v u_2 + 2z^{-1}\hat{f}_+(u_1 + u_2) = 2\epsilon\left[R_2 - z^{-1}\hat{f}_+(e)\right] \; , \tag{3.38}$$

and

$$\Box^2_v e = 5(R_1 + R_2) + \bar{p}_1 \cdot \Box R_2 + \bar{p}_2 \cdot \Box R_1 + \hat{h}(S) \; . \tag{3.39}$$

To obtain the differential equations for the functions v_1, v_2 and u_3, we must first simplify Eq.(2.68). This can be accomplished if we realize that

$$\Box^2 \partial^\nu C^\lambda_{\mu\nu} = \epsilon \partial^\nu D^\lambda_{\mu\nu} \; , \tag{3.40}$$

which follows from a direct application of ∂^ν to Eq. (2.68). This implies that

$$\epsilon\, \Box^{-2} \partial^\nu D^\lambda_{\mu\nu} = \partial^\nu C^\lambda_{\mu\nu} \; , \tag{3.41}$$

and therefore Eq.(2.68) becomes

$$\Box^2 C^\lambda_{\mu\nu} + 2(\partial_\mu \partial^\rho C^\lambda_{\nu\rho} - \partial_\nu \partial^\rho C^\lambda_{\mu\rho}) = -\epsilon D^\lambda_{\mu\nu} \ . \tag{3.42}$$

Using the definition of $C^\lambda_{\mu\nu}$ given in Eq.(2.78), we can write $C^\lambda_{\alpha\rho}$ as

$$C^\lambda_{\alpha\rho} = (p_{1\rho} v_2 + p_{2\rho} v_1)\tilde{\delta}^\lambda_\alpha - (p_{1\alpha} v_2 + p_{2\alpha} v_1)\tilde{\delta}^\lambda_\rho$$

$$+ u_3 (p_{1\alpha} p_{2\rho} - p_{2\alpha} p_{1\rho})\tilde{p}^\lambda \ . \tag{3.43}$$

Applying ∂^ρ to Eq.(3.42) gives

$$\partial^\rho C^\lambda_{\alpha\rho} = [3(v_1 + v_2) + \bar{p}_1 \cdot \Box v_2 + \bar{p}_2 \cdot \Box v_1]\, \tilde{\delta}^\lambda_\alpha - (p_{1\alpha} \partial^\lambda v_2 + p_{2\alpha} \partial^\lambda v_1)$$

$$+ [p_{1\alpha} \bar{p}_2 \cdot \Box u_3 - p_{2\alpha} \bar{p}_1 \cdot \Box u_3]\, \tilde{p}^\lambda + 4u_3(p_{1\alpha} - p_{2\alpha})\tilde{p}^\lambda \ . \tag{3.44}$$

In general, if we apply ∂^α to $C^\lambda_{\mu\nu}$, this results in the identity

$$\begin{aligned} \partial^\alpha C^\lambda_{\mu\nu} = \ & \delta^\alpha_\nu (v_1 + v_2) + (p_{1\nu} \partial^\alpha v_2 + p_{2\nu} \partial^\alpha v_1)\tilde{\delta}^\lambda_\mu \\ & - \delta^\alpha_\mu (v_1 + v_2) + (p_{1\mu} \partial^\alpha v_2 + p_{2\mu} \partial^\alpha v_1)\tilde{\delta}^\lambda_\nu \\ & + (p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})\tilde{p}^\lambda \partial^\alpha u_3 + u_3 (p_{1\mu} p_{2\nu} - p_{2\mu} p_{1\nu})\delta^{\lambda\alpha} \\ & + u_3 \left[\delta^\alpha_\mu p_{2\nu} + \delta^\alpha_\nu p_{1\mu} - \delta^\alpha_\mu p_{1\nu} + \delta^\alpha_\nu p_{2\mu}\right] \tilde{p}^\lambda \ . \end{aligned} \tag{3.45}$$

Applying ∂_α to Eq.(3.45) and factoring all scalar functions multiplying the tensors $(p_{1\nu}\tilde{\delta}^\lambda_\mu - p_{1\mu}\tilde{\delta}^\lambda_\nu), (p_{2\nu}\tilde{\delta}^\lambda_\mu - p_{2\mu}\tilde{\delta}^\lambda_\nu)$, and $(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{p}^\lambda$ gives

$$\begin{aligned} \Box^2 C^\lambda_{\mu\nu} = & (p_{1\nu}\tilde{\delta}^\lambda_\mu - p_{1\mu}\tilde{\delta}^\lambda_\nu)\{\Box^2_s v_2 + 2z^{-1}\hat{f}_+(v_1 + v_2) - 2u_3\} \\ & + (p_{2\nu}\tilde{\delta}^\lambda_\mu - p_{2\mu}\tilde{\delta}^\lambda_\nu)\{\Box^2_s v_1 + \quad 2z\hat{f}_-(v_1 + v_2) + 2u_3\} \\ & + (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})\tilde{p}^\lambda\{\Box^2_s u_3 + 6\hat{h}(u_3)\} \ . \end{aligned} \tag{3.46}$$

An analogous, but otherwise tedious calculation yields

$$\begin{aligned}\partial_\mu \partial^\rho C^\lambda_{\nu\rho} - \partial_\nu \partial^\rho C^\lambda_{\mu\rho} = \quad & (p_{1\nu}\tilde{\delta}^\lambda_\mu - p_{1\mu}\tilde{\delta}^\lambda_\nu) G_1(v_1, v_2, u_3) \\ & + (p_{2\nu}\tilde{\delta}^\lambda_\mu - p_{2\mu}\tilde{\delta}^\lambda_\nu) G_2(v_1, v_2, u_3) \\ & + \tilde{p}^\lambda (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu}) G_3(v_1, v_2, u_3) \,, \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} G_1(v_1,v_2,u_3) = & \Big\{ -3z^{-1}\hat{f}_+(v_1+v_2) - 2\hat{h}(v_2) \\ & - z^{-1}\hat{f}_+[\bar{p}_1 \cdot \Box v_2] - z^{-1}\hat{f}_+[\bar{p}_2 \cdot \Box v_1] + \bar{p}_2 \cdot \Box u_3 + 4u_3 \Big\} \,, \\ G_2(v_1,v_2,u_3) = & \Big\{ - \quad 3z\hat{f}_-(v_1+v_2) - 2\hat{h}(v_1) \\ & - \quad z\hat{f}_-[\bar{p}_1 \cdot \Box v_2] - \quad z\hat{f}_-[\bar{p}_2 \cdot \Box v_1] - \bar{p}_1 \cdot \Box u_3 - 4u_3 \Big\} \,, \\ G_3(v_1,v_2,u_3) = & \Big\{ \tfrac{1}{2}\hat{h}[\hat{h}(v_2 - v_1)] - \bar{p} \cdot \Box u_3 - 7\hat{h}(u_3) \Big\} \,. \end{aligned} \tag{3.48}$$

Substituting Eqs.(3.46) and (3.47) into Eq.(3.42) gives

$$\begin{aligned}\Box^2 C^\lambda_{\mu\nu} + 2(\partial_\mu \partial^\rho C^\lambda_{\nu\rho} - \partial_\nu \partial^\rho C^\lambda_{\mu\rho}) = & (p_{1\nu}\tilde{\delta}^\lambda_\mu - p_{1\mu}\tilde{\delta}^\lambda_\nu) H_1(v_1, v_2, u_3) \\ & + (p_{2\nu}\tilde{\delta}^\lambda_\mu - p_{2\mu}\tilde{\delta}^\lambda_\nu) H_2(v_1, v_2, u_3) \\ & + \tilde{p}^\lambda (p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu}) H_3(v_1, v_2, u_3) \,, \end{aligned} \tag{3.49}$$

where

$$\begin{aligned} H_1(v_1,v_2,u_3) = & \Big\{ \Box^2_v v_2 - 4z^{-1}\hat{f}_+(v_1+v_2) - 4\hat{h}(v_2) \\ & - 2z^{-1}\hat{f}_+[\bar{p}_1 \cdot \Box v_2] - 2z^{-1}\hat{f}_+[\bar{p}_2 \cdot \Box v_1] + 2\bar{p}_2 \cdot \Box u_3 + 6u_3 \Big\} \,, \\ H_2(v_1,v_2,u_3) = & \Big\{ \Box^2_v v_1 - \quad 4z\hat{f}_-(v_1+v_2) - 4\hat{h}(v_1) \\ & - \quad 2z\hat{f}_-[\bar{p}_1 \cdot \Box v_2] - \quad 2z\hat{f}_-[\bar{p}_2 \cdot \Box v_1] - 2\bar{p}_1 \cdot \Box u_3 - 6u_3 \Big\} \,, \\ H_3(v_1,v_2,u_3) = & \Big\{ \Box^2_v u_3 - 10\hat{h}(u_3) - 2\bar{p} \cdot \Box u_3 + \hat{h}[\hat{h}(v_2 - v_1)] \Big\} \,. \end{aligned} \tag{3.50}$$

Comparing Eqs.(3.42), (3.50) with Eq.(2.83) and factoring all terms proportional to the tensors $(p_{1\nu}\tilde{\delta}^{\lambda}_{\mu} - p_{1\mu}\tilde{\delta}^{\lambda}_{\nu})$, $(p_{2\nu}\tilde{\delta}^{\lambda}_{\mu} - p_{2\mu}\tilde{\delta}^{\lambda}_{\nu})$ and $\tilde{p}^{\lambda}(p_{1\mu}p_{2\nu} - p_{2\mu}p_{1\nu})$ yields the desired scalar differential equations for the functions v_1, v_2 and u_3. They are

$$\begin{aligned}\Box^2_v v_1 - 4\hat{h}(v_1) - \; & 2z\hat{f}_-[\bar{p}_1 \cdot \Box v_2] - \; 2z\hat{f}_-[\bar{p}_2 \cdot \Box v_1] \\ - \; & 2z\hat{f}_-(v_1 + v_2) - 6u_3 - 2\bar{p}_1 \cdot \Box u_3 = 2\epsilon \; S_1 \; , \end{aligned} \tag{3.51}$$

$$\begin{aligned}\Box^2_v v_2 - 4\hat{h}(v_2) - 2z^{-1}\hat{f}_+[\bar{p}_1 \cdot \Box v_2] - 2z^{-1}\hat{f}_+[\bar{p}_2 \cdot \Box v_1] \\ - 2z^{-1}\hat{f}_+(v_1 + v_2) + 6u_3 + 2\bar{p}_2 \cdot \Box u_3 = 2\epsilon \; S_2 \; , \end{aligned} \tag{3.52}$$

$$\Box^2_v u_3 - 10\hat{h}(u_3) - 2\bar{p} \cdot \Box\hat{h}(u_3) + \hat{h}\left[\hat{h}(v_2 - v_1)\right] = -\epsilon \; R_3 \; , \tag{3.53}$$

where the functions S_1, S_2 and R_3 in the right-hand side of Eqs. (3.51) to (3.53) are given by Eqs.(2.88), (2.90) and (2.91).

To obtain the final differential equation for the function v_3, we first simplify Eq.(2.69) by realizing that

$$\Box^2 \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = -2\epsilon \partial^{\rho} D^{\lambda}_{\mu\nu\rho} \; . \tag{3.54}$$

This equation still contains all the information of the original equation. It is possible to reduce this equation to a vector equation by multiplying Eq.(3.54) with the tensor $p_1^{\mu} p_2^{\nu}$. Due to the antisymmetry of $C^{\lambda}_{\mu\nu\rho}$ and $D^{\lambda}_{\mu\nu\rho}$ under the interchange of two of its indices, and the fact that $C^{\lambda}_{\mu\nu\rho} = 0$ if any two indices are equal, we can solve the equivalent equation

$$\Box^2 p_1^{\mu} p_2^{\nu} \partial^{\rho} C^{\lambda}_{\mu\nu\rho} = \Box^2 \partial^{\rho} \left[p_1^{\mu} p_2^{\nu} C^{\lambda}_{\mu\nu\rho}\right] = -2\epsilon \partial^{\rho} \left[p_1^{\mu} p_2^{\nu} D^{\lambda}_{\mu\nu\rho}\right] \; . \tag{3.55}$$

Using Eq.(2.79) and contracting $C^{\lambda}_{\mu\nu\rho}$ with $p_1^{\mu} p_2^{\nu}$ gives

$$p_1^{\mu} p_2^{\nu} C^{\lambda}_{\mu\nu\rho} = v_3 \Big\{ (\bar{p}_1 \cdot \bar{p}_2 p_{2\rho} - \bar{p}_2^2 p_{1\rho})\tilde{p}^{\lambda} + (\bar{p}_1 \cdot \bar{p}_2 p_{1\rho} - \bar{p}_1^2 p_{2\rho})\tilde{p}^{\lambda}$$

$$+[\bar{p}_1^2\bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2]\tilde{\delta}_\rho^\lambda \Big\} \; . \tag{3.56}$$

Applying ∂^ρ to Eq.(3.56) gives

$$\partial^\rho \left[p_1^\mu p_2^\nu C^\lambda_{\mu\nu\rho}\right] = -8\bar{k}^2 v_3 \tilde{p}^\lambda \; . \tag{3.57}$$

Similarly, from Eq.(2.84) it follows that

$$p_1^\mu p_2^\nu D^\lambda_{\mu\nu\rho} = S_3 \Big\{ (\bar{p}_1 \cdot \bar{p}_2 p_{2\rho} - \bar{p}_2^2 p_{1\rho})\tilde{p}^\lambda + (\bar{p}_1 \cdot \bar{p}_2 p_{1\rho} - \bar{p}_1^2 p_{2\rho})\tilde{p}^\lambda$$

$$+[\bar{p}_1^2\bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2]\tilde{\delta}_\rho^\lambda \Big\} \; , \tag{3.58}$$

and

$$\partial^\rho \left[p_1^\mu p_2^\nu D^\lambda_{\mu\nu\rho}\right] = -8\bar{k}^2 S_3 \tilde{p}^\lambda \; . \tag{3.59}$$

Thus, the function v_3 must satisfy the differential equation

$$\Box_v^2 v_3 = -2\epsilon S_3 \; . \tag{3.60}$$

3-3 The Model Equation in the Asymptotic Region

To obtain the solutions to the differential equations in the asymptotic region, we must be very careful how to approximate the functions $R, R_1, \ldots, S, S_1, \ldots$. In the inner asymptotic region, we *cannot* neglect those terms multiplied by the electron propagator functions A_1 and A_2 unless they are added (substracted) from terms that are clearly larger. For example we can approximate $A_1 A_2 + \bar{p}_1 \cdot \bar{p}_2 \sim \bar{p}_1 \cdot \bar{p}_2$ but *cannot* neglect the term $(A_1 + A_2)v$ against $\bar{p}_1 \cdot \bar{p}_2 \; u$. If we do so, it *will not* be possible to join the asymptotic solutions to the perturbation solutions. These terms are not negligible since, in the inner asymptotic region, the form of the electron functions is

$$\begin{aligned} A_1 &\sim m(\quad xz/m^2)^{-\eta} \\ A_2 &\sim m(xz^{-1}/m^2)^{-\eta} \end{aligned} \tag{3.61}$$

where by Eq.(2.38), $\eta = 3\epsilon/4 \approx 1.75 \times 10^{-3}$. Although the functions A_1 and A_2 approach zero for large values of x, one has go to an extremely high momentum to see an appreciable difference. This problem does not arise in the first five since *all* terms containing the electron propagator functions are negligible compare to those retained. In the inner asymptotic region the following form for the functions R_1, R_2, S, S_3 are correct:

$$R_1 \sim \left[v + \bar{p}_1^2 u_2 - (\bar{p}_1^2 - \bar{p}_1 \cdot \bar{p}_2)v_3\right]/D \ , \tag{3.62}$$

$$R_2 \sim \left[v + \bar{p}_2^2 u_1 - (\bar{p}_2^2 + \bar{p}_1 \cdot \bar{p}_2)v_3\right]/D \ , \tag{3.63}$$

$$S \sim \left[-\bar{p}_1 \cdot \bar{p}_2 v + \left[\bar{p}_1^2 \bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2\right]v_3\right]/ \ , D \tag{3.64}$$

$$S_3 \sim \left[v + \bar{p}_1 \cdot \bar{p}_2 v_3\right]/D \ . \tag{3.65}$$

The remaining functions have the form

$$R \sim \left[\bar{p}_1 \cdot \bar{p}_2 u + (A_1\bar{p}_2 + A_2\bar{p}_1) \cdot (u_1\bar{p}_2 + u_2\bar{p}_1) + (A_1 + A_2)v \right.$$
$$\left. + (\bar{p}_1^2 - \bar{p}_1 \cdot \bar{p}_2)v_2 - (\bar{p}_2^2 - \bar{p}_1 \cdot \bar{p}_2)v_1 - [\bar{p}_1^2\bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2]u_3\right]/D \ . \tag{3.66}$$

$$R_3 \sim \left[-u - A_1u_2 - A_2u_1 + \bar{p}_1 \cdot \bar{p}_2 u_3 \right.$$
$$\left. + (v_2 - v_1) + (A_1 + A_2)v_3\right]/D \ . \tag{3.67}$$

$$S_1 \sim \left[-A_1 v - \bar{p}_1^2 v_2 - (A_2\bar{p}_1^2 + A_1\bar{p}_1 \cdot \bar{p}_2)v_3\right]/D \ . \tag{3.68}$$

$$S_2 \sim \left[\ A_2 v - \bar{p}_2^2 v_1 + (A_1\bar{p}_2^2 + A_2\bar{p}_1 \cdot \bar{p}_2)v_3\right]/D \ . \tag{3.69}$$

At this point, it is important to realize that in the asymptotic region the system of nine coupled differential equations has decoupled into two set of equations. Notice that Eqs.(3.62) to (3.65) are coupled together but not to the remaining four. Therefore, it is possible to attempt to solve this set of five equations independently from the rest. This is the approach we will follow.

It is possible to further simplify the differential equations. To this end an explicit form for the coefficient of the five functions v, u_1, u_2, v_3, e will be used.

From Eqs.(3.4), (3.5) and (2.71) it follows that the denominator D that appears in the right-hand side of all the differential equations can be approximated by the expression

$$D \sim x^2 \; . \tag{3.70}$$

This approximation is valid in both the inner and outer asymptotic regions provided that $(x/m^2) \gg y$. If we realize that in the asymptotic region, $\bar{p}^2 \sim \bar{p}_1 \cdot \bar{p}_2 \sim xy$, then the differential equations are somewhat simplified. For convenience, all nine asymptotic forms of the differential equations are presented in Table 3-1.

Table 3-1 Asymptotic Forms of Differential Equations

$$\Box_v^2 v - 2\hat{h}(v) + 2(u_1 + u_2) = 2\epsilon(S - e) \tag{3.71}$$

$$\Box_v^2 u_1 + \quad 2z\hat{f}_-(u_1 + u_2) = 2\epsilon\big[R_1 - \quad z\hat{f}_-(e)\big] \tag{3.72}$$

$$\Box_v^2 u_2 + 2z^{-1}\hat{f}_+(u_1 + u_2) = 2\epsilon\big[R_2 - z^{-1}\hat{f}_+(e)\big] \tag{3.73}$$

$$\Box_v^2 v_3 = -2\epsilon S_3 \tag{3.74}$$

$$\Box_v^2 e = 5(R_1 + R_2) + \bar{p}_1 \cdot \Box R_2 + \bar{p}_2 \cdot \Box R_1 + \hat{h}(S) \tag{3.75}$$

$$\Box_v^2 u = -3\epsilon R \tag{3.76}$$

$$\begin{aligned} \Box_v^2 v_1 - 4\hat{h}(v_1) - \quad 2z\hat{f}_-[\bar{p}_1 \cdot \Box v_2] - \quad 2z\hat{f}_-[\bar{p}_2 \cdot \Box v_1] \\ - \quad 2z\hat{f}_-(v_1 + v_2) - 6u_3 - 2\bar{p}_1 \cdot \Box u_3 = 2\epsilon\ S_1 \end{aligned} \tag{3.77}$$

$$\begin{aligned} \Box_v^2 v_2 - 4\hat{h}(v_2) - 2z^{-1}\hat{f}_+[\bar{p}_1 \cdot \Box v_2] - 2z^{-1}\hat{f}_+[\bar{p}_2 \cdot \Box v_1] \\ - 2z^{-1}\hat{f}_+(v_1 + v_2) + 6u_3 + 2\bar{p}_2 \cdot \Box u_3 = 2\epsilon\ S_2 \end{aligned} \tag{3.78}$$

$$\Box_v^2 u_3 - 10\hat{h}(u_3) - 2\bar{p} \cdot \Box\hat{h}(u_3) + \hat{h}\big[\hat{h}(v_2 - v_1)\big] = -\epsilon\ R_3\ . \tag{3.79}$$

where

$$\begin{aligned} R \sim \frac{1}{x^2}\big[xy\ u + (A_1\bar{p}_2 + A_2\bar{p}_1) \cdot (u_1\bar{p}_2 + u_2\bar{p}_1) + (A_1 + A_2)v \\ + x\sqrt{y^2 - 1}\,(v_1 + v_2) + x^2(y^2 - 1)u_3\big] \end{aligned} \tag{3.80}$$

$$R_1 \sim \frac{1}{x^2}\left[v + \ xz \quad u_2 - x\sqrt{y^2 - 1}\ \ v_3\right] \tag{3.81}$$

$$R_2 \sim \frac{1}{x^2}\left[v + xz^{-1}\ u_1 + x\sqrt{y^2 - 1}\ \ v_3\right] \tag{3.82}$$

$$R_3 \sim \frac{1}{x^2}\big[-u - A_1 u_2 - A_2 u_1 + xy\ u_3 + (v_2 - v_1) + (A_1 + A_2)v_3\big] \tag{3.83}$$

$$S \sim -\frac{1}{x}\ [-y\ v + x(y^2 - 1)\ v_3] \tag{3.84}$$

$$S_1 \sim \frac{1}{x^2}\big[-A_1 v - xzv_2 - x(zA_2 + yA_1)v_3\big] \tag{3.85}$$

$$S_2 \sim \frac{1}{x^2}\big[\ \ A_2 v - xz^{-1}v_1 + x(z^{-1}A_1 + y\ A_2)v_3\big] \tag{3.86}$$

$$S_3 \sim \frac{1}{x^2}\ [v + xy\ v_3] \tag{3.87}$$

CHAPTER IV
ASYMPTOTIC SOLUTIONS TO THE MODEL EQUATION

We seek solutions to the differential equations given in Eqs.(3.61) to (3.69) which join smoothly with the perturbation solutions. In order to do this, we use as a guide, the asymptotic form of the perturbation solutions found in Table C-2 of Appendix C. As can be seen from this table, all perturbative solutions can be written in the form of a power law in the variable x. With this in mind, we make the hypothesis that the asymptotic solutions to the differential equations are of the form

$$W^\lambda = x^\alpha \bar{g}(y)\tilde{p}^\lambda \; , \tag{4.1}$$

where, at this stage, we will use a bar over functions to denote functions that are only y-dependent. The justification for this assumption lies in the fact that the asymptotic expression for the perturbation solutions are of this form and they satisfy the model equation in this region (within .5%) with the exception of the v_1,v_2 and u_3 functions. The latter are good only within 3%. When we apply the D'Alambertian operator to such functions, it is clear that

$$\begin{aligned}\Box^2 W^\lambda = \Bigg\{ & \frac{4\bar{k}^2}{x^2}\left[(y^2-1)\frac{d^2\bar{g}}{dy^2} + y\frac{d\bar{g}}{dy} - \alpha^2\bar{g}\right] \\ & - \frac{4(\alpha+2)}{x}\left[(y^2-1)\frac{d\bar{g}}{dy} - \alpha y\,\bar{g}\right]\Bigg\} x^\alpha \tilde{p}^\lambda \; . \end{aligned} \tag{4.2}$$

It is evident from this equation that, if $\bar{g}(y)$ is a smooth function of the variable y, then for large values of x, it is possible to neglect the first term since it is one power of x smaller than the second one. This leads to the approximate form of Eq.(4.2)

$$\Box^2 W^\lambda \sim x^\alpha \tilde{p}^\lambda \left\{ \frac{4(\alpha+2)}{x} \hat{D}_\alpha \bar{g} \right\} \tag{4.3}$$

where

$$\hat{D}_\alpha = -(y^2-1)\frac{d}{dy} + \alpha y \ . \tag{4.4}$$

To simplify the differential equations in the asymptotic region, it is necessary to find the form of the operators $\bar{p}_1 \cdot \Box$, $\bar{p}_2 \cdot \Box$, $\hat{h}$, $\hat{l}$, etc., when acting upon functions of the form given by Eq.(4.1). It is easy to verify that when these operators are applied to functions of the form $\Phi = x^\alpha \bar{g}(y)$, the following relations are valid:

$$\bar{p}_1 \cdot \Box\Phi = x^\alpha \left[z\hat{D}_\alpha + \alpha \right] \bar{g} \ , \tag{4.5}$$

$$\bar{p}_2 \cdot \Box\Phi = x^\alpha \left[z^{-1}\hat{D}_\alpha + \alpha \right] \bar{g} \ , \tag{4.6}$$

$$\bar{p} \cdot \Box\Phi = x^\alpha \left[y\hat{D}_\alpha + \alpha \right] \bar{g} \ , \tag{4.7}$$

$$\bar{k} \cdot \Box\Phi = x^\alpha \left[\sqrt{y^2-1}\,\hat{D}_\alpha \right] \bar{g} \ , \tag{4.8}$$

$$\hat{h}(\Phi) = 2x^{\alpha-1}\hat{D}_\alpha \bar{g} \ , \tag{4.9}$$

$$\hat{l}(\Phi) = -2x^{\alpha-1}\frac{1}{\sqrt{y^2-1}} \left[y\hat{D}_\alpha - \alpha \right] \bar{g} \ , \tag{4.10}$$

$$\hat{f}_+(\Phi) = -\frac{x^{\alpha-1}}{\sqrt{y^2-1}}(\hat{D}_\alpha - \alpha z)\bar{g} \ , \tag{4.11}$$

$$\hat{f}_-(\Phi) = +\frac{x^{\alpha-1}}{\sqrt{y^2-1}}(\hat{D}_\alpha - \alpha z^{-1})\bar{g} \ . \tag{4.12}$$

Also, a combined application of the $\hat{h}$ and $\hat{l}$ operators upon these type of functions satisfies the following relations:

$$\hat{h}[\hat{l}(\Phi)] = \hat{l}[\hat{h}(\Phi)] \ , \tag{4.13}$$

$$\hat{h}[\bar{p} \cdot \Box\Phi] = 2\hat{h}(\Phi) + \bar{p} \cdot \Box\hat{h}(\Phi) \ , \tag{4.14}$$

$$\hat{h}[\bar{k} \cdot \Box\Phi] = \bar{k} \cdot \Box\hat{h}(\Phi) \tag{4.15}$$

where

$$\hat{h}[\bar{k} \cdot \Box\Phi] = 2x^{\alpha-1}\sqrt{y^2-1}\,[\hat{D}_\alpha^2 - y\,\hat{D}_\alpha]\bar{g} \ , \tag{4.16}$$

$$\hat{h}[\bar{p}\cdot\Box\Phi] = 2x^{\alpha-1}\left[y\hat{D}^2_\alpha + \left[\alpha - (y^2-1)\right]\hat{D}_\alpha\right]\bar{g} \ , \tag{4.17}$$

$$\hat{l}[\bar{k}\cdot\Box\Phi] = 2x^{\alpha-1}\left[y\ \hat{D}^2_\alpha - (y^2+\alpha)\hat{D}_\alpha\right]\bar{g} \ , \tag{4.18}$$

$$\hat{l}[\hat{h}(\Phi)] = -4\frac{x^{\alpha-2}}{\sqrt{y^2-1}}\left[y\ \hat{D}^2_\alpha - \left[(y^2-1)+\alpha\right]\hat{D}_\alpha\right]\bar{g} \ . \tag{4.19}$$

and the operator $\hat{D}^2_\alpha$ is defined by the equation

$$\hat{D}^2_\alpha = (y^2-1)^2\frac{d^2}{dy^2} - 2(\alpha-1)y(y^2-1)\frac{d}{dy} + \alpha\left[1+(\alpha-1)y^2\right] \ . \tag{4.20}$$

Using Table C-2 of Appendix C, it is reasonable to assume the following forms for the functions v, u_1, u_2, v_3 and e:

$$v = c_1 x^{\alpha+1}\bar{v} \ , \tag{4.21}$$

$$u_1 = -u_2 = x^\alpha \bar{u}_1 \ , \tag{4.22}$$

$$v_3 = x^\alpha \bar{v}_3 \ , \tag{4.23}$$

and

$$e = x^\alpha \bar{e} \tag{4.24}$$

where c_1 is an arbitrary constant. Furthermore, the asymptotic behavior of the v-function (see Appendix D) leads to the conclusion that the value of α must be equal to -1. Other solutions are possible with $\alpha \neq -1$ but they proved to vanish at large values of the variable x. Such solutions are not favored since they do not have an asymptotic behavior consistent with Appendix D and they do not join smoothly with the perturbation formulas found in Eqs.(C.41) to (C.46). To find the asymptotic solutions to equations (3.71) to (3.75) that join the perturbation forms, we set all functions in the right-hand side of the differential equations equal to zero except for the v and e-functions. Substituting Eqs.(4.21) to (4.24) into Eqs.(3.71) to (3.75) and setting $\alpha = -1$ gives

$$4[\hat{D}_{-1} + y]\bar{v} = -2\epsilon[y\bar{v} + \bar{e}] \ , \tag{4.25}$$

$$4\hat{D}_{-1}\bar{u}_1 = 2\epsilon[\bar{v} + z(\bar{e} + \sqrt{y^2-1}\,\bar{e}')] \ , \tag{4.26}$$

$$4\hat{D}_{-1}\bar{u}_2 = 2\epsilon[\bar{v} + z^{-1}(\bar{e} - \sqrt{y^2-1}\,\bar{e}')] \ , \tag{4.27}$$

$$4\hat{D}_{-1}\bar{v}_3 = -2\epsilon\bar{v} \ , \tag{4.28}$$

and

$$4\hat{D}_{-1}\bar{e} = 10\bar{v}+\Big\{ \quad z(\hat{D}_{-1} - y) - 2\Big\}\,\bar{v}+\Big\{z^{-1}(\hat{D}_{-1} - y) - 2\Big\}\,\bar{v}+2\hat{D}_{-1}(-y\bar{v}) \tag{4.29}$$

where

$$\hat{D}_{-1} = (1-y^2)\frac{d}{dy} - y \ . \tag{4.30}$$

The primes denote derivatives with respect to the variable y.

Notice that with the choice, $\alpha = -1$, all five functions are in agreement with the x-dependency of the perturbation solutions and in effect have reduced the problem to that of finding the solutions to Eqs.(4.25) to (4.30) which are only y-dependent. To solve these equations, we start by adding Eqs.(4.26) and (4.27). This gives the result

$$2\hat{D}_{-1}(\bar{u}_1 + \bar{u}_2) = \epsilon(\bar{v} - \hat{D}_{-1}\bar{e}) \ . \tag{4.31}$$

Since we assume $\bar{u}_1 = -\bar{u}_2$ in obtaining these differential equations, we conclude that, for consistency,

$$\hat{D}_{-1}\bar{e} = \bar{v} \ . \tag{4.32}$$

Applying $\hat{D}_{-1}$ to Eq.(4.25) and substituting the previous expression gives

$$4\hat{D}_{-1}[\hat{D}_{-1} + y]\bar{v} = -2\epsilon[\hat{D}(y\bar{v}) + \bar{v}] \ . \tag{4.33}$$

Using the identity

$$\hat{D}\,y = y\hat{D} - (y^2-1) \ , \tag{4.34}$$

and moving all the terms to the left side of in Eq.(4.33) yields the result,

$$4\hat{D}_{-1}[\hat{D}_{-1} + y]\bar{v} + 2\epsilon[y\hat{D}_{-1}\bar{v} - (y^2 - 1)\bar{v}] + 2\epsilon\bar{v} = 0 \ . \tag{4.35}$$

In terms of the operator $\hat{D}^2_{-1}$ obtained from Eq.(4.20) by letting $\alpha = -1$,

$$\hat{D}^2_{-1}\bar{v} + (1 + \tfrac{1}{2}\epsilon)y\hat{D}_{-1}\bar{v} + [(1 - \tfrac{1}{2}\epsilon)(y^2 - 1) + \tfrac{1}{2}\epsilon]\bar{v} = 0 \ . \tag{4.36}$$

In terms of the derivatives of $\bar{v}$ with respect to y, Eq.(4.36) can be written as

$$(y^2 - 1)\bar{v}'' + (3 - \tfrac{1}{2}\epsilon)y\,\bar{v}' - \epsilon\bar{v} = 0 \ . \tag{4.37}$$

It is possible to transform Eq.(4.37) with the substitution,

$$\bar{v} = (y^2 - 1)^{\tau}\bar{\bar{v}} \ , \tag{4.38}$$

$$\bar{v}' = (y^2 - 1)^{\tau}\,\bar{\bar{v}}' + 2\tau y(y^2 - 1)^{\tau - 1}\,\bar{\bar{v}} \ , \tag{4.39}$$

and

$$\bar{v}'' = (y^2-1)^{\tau}\bar{\bar{v}}'' + 4\tau y(y^2-1)^{\tau-1}\bar{\bar{v}}' + \{2\tau(y^2-1)^{\tau-1} + 4\tau(\tau-1)y^2(y^2-1)^{\tau-2}\}\bar{\bar{v}} \ . \tag{4.40}$$

Substituting Eqs.(4.38) to (4.40) into Eq.(4.37) gives

$$\begin{aligned}(y^2 - 1)^{\tau}\Big\{&(y^2 - 1)\,\bar{\bar{v}}'' + [4\tau + (3 - \tfrac{1}{2}\epsilon)]y\,\bar{\bar{v}}' \\ &+ \big[2\tau + [\,4\tau(\tau - 1) + 2\tau(3 - \tfrac{1}{2}\epsilon)\,]y^2(y^2 - 1)^{-1} - \epsilon\big]\,\bar{\bar{v}}\Big\} = 0 \ .\end{aligned} \tag{4.41}$$

To simplify this equation, we select the value of the constant τ to satisfy the condition

$$4\tau + (3 - \tfrac{1}{2}\epsilon) = 2 \tag{4.42}$$

or solving for τ

$$\tau = \tfrac{1}{4}(-1 + \tfrac{1}{2}\epsilon) \ . \tag{4.43}$$

Making these substitutions in Eq.(4.41) gives

$$(y^2-1)\,\bar{\bar{v}}''+2y\,\bar{\bar{v}}'+\{2\tau+[4\tau(\tau-1)-2\tau(4\tau-2)]y^2(y^2-1)^{-1}-\epsilon\}\bar{\bar{v}}=0\ , \quad (4.44)$$

or after some simplifications

$$(y^2-1)\,\bar{\bar{v}}''+2y\,\bar{\bar{v}}'-\{(4\tau^2-2\tau+\epsilon)+\frac{4\tau^2}{(y^2-1)}\}\,\bar{\bar{v}}=0\ . \quad (4.45)$$

If we define the constant μ such that $\mu=\pm 2\tau$ and the constant ν by the relation

$$\nu(\nu+1)=4\tau^2-2\tau+\epsilon\ , \quad (4.46)$$

then Eq.(4.45) can be written in the form

$$(y^2-1)\,\bar{\bar{v}}''+2y\,\bar{\bar{v}}'-\{\nu(\nu+1)+\frac{\mu^2}{(y^2-1)}\}\,\bar{\bar{v}}=0\ , \quad (4.47)$$

where

$$\mu=\pm\tfrac{1}{2}(-1+\tfrac{1}{2}\epsilon) \qquad \text{and} \qquad \nu=\tfrac{1}{2}(1+\tfrac{1}{2}\epsilon)\ . \quad (4.48)$$

The general solution to Eq.(4.47) can be expressed in terms of the Associated Legendre functions $P_\nu^\mu(z)$(See Appendix E).[28,29] These, in turn, can be represented by the hypergeometric series $F(a,b,c;y)$ by the identity,

$$P_\nu^\mu(y)=2^\mu\frac{(y^2-1)^{-\frac{1}{2}\mu}}{\Gamma(1-\mu)}y^{\mu+\nu}F(\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,1-\mu;1-y^{-2})\ , \quad (4.49)$$

valid for $y\geq 1$. For computational purposes, however, this expresion is not convenient due to the slow convergence of the hypergeometric series when its argument is near 1. For large values of y the following expression gives a rapid convergence of the series:

$$\begin{aligned}P_\nu^\mu(y)=&\ \frac{2^{-(\nu+1)}}{\sqrt{\pi}}\frac{\Gamma(-\frac{1}{2}-\nu)}{\Gamma(-\mu-\nu)}y^{\mu-\nu-1}(y^2-1)^{-\frac{1}{2}\mu}\\ &\times F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\nu+\tfrac{3}{2};y^{-2})\\ &+\frac{2^\nu}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu-\mu)}y^{\mu+\nu}(y^2-1)^{-\frac{1}{2}\mu}\\ &\times F(-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu,\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\tfrac{1}{2}-\nu;y^{-2})\ . \quad (4.50)\end{aligned}$$

The $\bar{v}$-function can then be written in the form

$$v = c_1(y^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(y) \; . \tag{4.51}$$

where μ and ν are defined in Eq.(4.48). We now turn to the problem of solving Eq.(4.32) with the solution to the $\bar{v}$-function differential equation in the right-hand side. From Eq.(4.32) and the expression for $\hat{D}_{-1}$ given in Eq.(4.30), the differential equation that the $\bar{e}$-function must satisfy becomes

$$(y^2-1)\,\bar{e}\,' + y\,\bar{e} = -c_1(y^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(y). \tag{4.52}$$

Let assume that this function has the same form as the $\bar{v}$-function, *i.e.*,

$$\bar{e} = -c_1(y^2-1)^{\lambda} P_\beta^\alpha(y) \tag{4.53}$$

and

$$\bar{e}\,' = -2\lambda y(y^2-1)^{\lambda-1} P_\beta^\alpha(y) - (y^2-1) P_\beta^{\alpha}{}'(y) \; . \tag{4.54}$$

Substituting these two equations into the preceding one gives

$$(y^2-1)^\lambda \left\{ (2\lambda+1) y P_\beta^\alpha(y) + (y^2-1) P_\beta^{\alpha}{}'(y) \right\} = (y^2-1)^{\frac{1}{2}\mu} P_\nu^\mu(y) \; , \tag{4.55}$$

from which it follows that $\lambda = \frac{1}{2}\mu$. With this substitution, this equation becomes

$$(\mu+1)y\, P_\beta^\alpha(y) + (y^2-1) P_\beta^{\alpha}{}'(y) = P_\nu^\mu(y) \; . \tag{4.56}$$

Using the identity,

$$(y^2-1) P_\beta^{\alpha}{}'(y) \equiv \beta\, y\, P_\beta^\alpha(y) - (\alpha+\beta) P_{\beta-1}^\alpha(y) \; , \tag{4.57}$$

yields the equation

$$(\mu+\beta+1)y\, P_\beta^\alpha(y) - (\alpha+\beta) P_{\beta-1}^\alpha(y) = P_\nu^\mu(y) \; . \tag{4.58}$$

If we set $\alpha = \mu$, and $\beta = -(1+\mu)$ then it follows that the first term in this equation vanishes and the conditions,

$$-(\alpha+\beta) = 1 \qquad \text{and} \quad \beta - 1 = -(\nu+1) \ , \tag{4.59}$$

are satisfied. It is evident from Eq.(E.17) that the e-function is given by the equation

$$\bar{e} = -c_1(y^2-1)^{\frac{1}{2}\mu} P^{\mu}_{-(1+\mu)}(y) = -c_1(y^2-1)^{\frac{1}{2}\mu} P^{\mu}_{\mu}(y) \ . \tag{4.60}$$

We now turn to the problem of solving Eq.(4.26) to find the function $\bar{u}_1$. To do this, we will transform this equation with the help of Eq.(4.12). From this equation, (with $\alpha = -1$), it follows that

$$z(\bar{e} + \sqrt{y^2-1}\,\bar{e}\,') = -\frac{z}{\sqrt{y^2-1}}(\hat{D}_{-1} + z^{-1})\bar{e} \ . \tag{4.61}$$

Substituting Eq.(4.32) into this equation and simplifying gives

$$z(\bar{e} + \sqrt{y^2-1}\,\bar{e}\,') = -\frac{z\,\bar{v} + \bar{e}}{\sqrt{y^2-1}} \ . \tag{4.62}$$

With this result, Eq.(4.26) yields the equation

$$4\hat{D}_{-1}\bar{u}_1 = -\frac{2\epsilon}{\sqrt{y^2-1}}[y\bar{v} + \bar{e}] \ , \tag{4.63}$$

or in terms of the first derivative of $\bar{u}_1$,

$$(y^2-1)\bar{u}_1' + y\bar{u}_1 = \frac{1}{2}\epsilon\frac{1}{\sqrt{y^2-1}}[y\bar{v} + \bar{e}] \ . \tag{4.64}$$

To solve this equation we make the substitution,

$$\bar{u}_1 = c_1(y^2-1)^{\frac{1}{2}(\mu-1)}\bar{\bar{u}}_1 \tag{4.65}$$

and

$$\bar{u}_1' = c_1(\mu-1)y\bar{\bar{u}}_1(y^2-1)^{\frac{1}{2}(\mu-3)} + (y^2-1)^{\frac{1}{2}(\mu-1)}(y^2-1)\bar{\bar{u}}_1{}' \ . \tag{4.66}$$

Then the left-hand side of the Eq. (4.64) becomes

$$(y^2-1)\bar{u}_1' + y\bar{u}_1 = (y^2-1)^{\frac{1}{2}(\mu-1)}\left[\mu\, y\bar{\bar{u}}_1 + (y^2-1)\,\bar{\bar{u}}_1{}'\right] . \tag{4.67}$$

The right-hand side of Eq.(4.62) can be written in the form,

$$\frac{1}{2}\epsilon\frac{1}{\sqrt{y^2-1}}[y\,\bar{v}+\bar{e}] = (y^2-1)^{\frac{1}{2}(\mu-1)}\left\{\frac{1}{2}\epsilon\left[yP_\nu^\mu(y) - P_\mu^\mu(y)\right]\right\} , \tag{4.68}$$

where we have substituted the expressions for $\bar{v}$ and $\bar{e}$ found in Eqs.(4.51) and (4.60). Equating the expressions given in Eqs.(4.67) and (4.68) yields the equation

$$(y^2-1)\,\bar{\bar{u}}_1{}' + \mu\, y\bar{\bar{u}}_1 = \frac{1}{2}\left[yP_\nu^\mu(y) - P_\mu^\mu(y)\right] \tag{4.69}$$

where μ and ν are defined in Eq.(4.48). Notice also, that from Eq.(4.48), $\nu - 1 = \mu$ and $\mu+\nu = \frac{1}{2}\epsilon$. Therefore Eq.(4.69) can be rewritten in the form,

$$(y^2-1)\,\bar{\bar{u}}_1{}' + \mu\, y\bar{\bar{u}}_1 = (\mu+\nu)\left[yP_\nu^\mu(y) - P_\mu^\mu(y)\right] . \tag{4.70}$$

It is possible to transform the right-hand side of the last equation by using several relations satisfied by the Legendre functions (see Appendix E; Eqs.(E.43) to (E.51)). It is straightforward to show that

$$\begin{aligned}(y^2-1)\bar{\bar{u}}_1{}' + \mu\, y\bar{\bar{u}}_1 &= (\nu-\mu+1)P_{\nu+1}^\mu(y) - 2yP_\nu^\mu(y)\\ &= (y^2-1)P_\nu^\mu(y)' + \mu\, yP_\nu^\mu(y) .\end{aligned} \tag{4.71}$$

It is easy to verify that the general solution to the homogeneous equation is given by the expression

$$(\bar{\bar{u}}_1)_h = c_2 P_{\mu-1}^\mu(y) \tag{4.72}$$

where c_2 is an arbitrary constant. It is evident from Eq.(4.71) that a particular solution to this equation is

$$(\bar{\bar{u}}_1)_p = P_\nu^\mu(y) . \tag{4.73}$$

If we set the constant $c_2 = -1$, then the general solution to Eq.(4.71) is

$$\bar{\bar{u}}_1 = P_\nu^\mu(y) - P_{\mu-1}^\mu(y) = (2\mu+1)\sqrt{y^2-1}\,P_\mu^{\mu-1}(y) \ . \tag{4.74}$$

Using Eqs.(4.65) and (4.66) we finally arrived at the desired solution

$$\bar{u}_1 = -\bar{u}_2 = c_1(2\mu+1)(y^2-1)^{\frac{1}{2}\mu}P_\mu^{\mu-1}(y) \ . \tag{4.75}$$

The $\bar{v}_3$-function can be easily obtained if we realize that the differential equation that it must satisfy is very similar to the differential equation satisfied by the $\bar{e}$-function. In fact, we notice that the difference between Eq.(4.28) and Eq.(4.32) is a constant factor. With this in mind, the function $\bar{v}_3$ can be expressed in terms of the $\bar{e}$-function in the form,

$$\bar{v}_3 = (\mu+\nu)\bar{e} = -(\mu+\nu)c_1(y^2-1)^{\frac{1}{2}\mu}P_\mu^\mu(y) \ . \tag{4.76}$$

The arbitrary constant c_1 needed to completely define the solutions can be determined by using the requirement that the v-function must approach the value

$$v \sim (1-\frac{3}{8}\epsilon) = (1-\frac{1}{2}\eta) \tag{4.77}$$

as $x \to \infty$ and $y \to 1$. This result was derived in Appendix D. From Eq.(4.51) and (E.57) we have that

$$\lim_{y\to 1} P_\nu^\mu(y) = \frac{2^\mu}{\Gamma(1-\mu)}(y^2-1)^{-\frac{1}{2}\mu} \tag{4.78}$$

and therefore

$$\lim_{y\to 1} v = c_1\frac{2^\mu}{\Gamma(1-\mu)} \ . \tag{4.79}$$

Equating the previous expression with Eq.(4.75) yields the result

$$c_1 = (1-\frac{1}{2}\eta)\frac{\Gamma(1-\mu)}{2^\mu} \ . \tag{4.80}$$

We now turn to the problem of solving Eq.(3.76) and (3.80) for the function $\bar{u}$. Its is possible to simplify the function R given in Eq.(3.80) in the following manner. If we expand the scalar product involving the functions $\bar{u}_1$ and $\bar{u}_2$ and substitute the condition that $\bar{u}_2 = -\bar{u}_1$, this equation becomes

$$R = \frac{1}{x^2}\Big\{xy\,u + u_1\left[A_1\bar{p}_1^2 + A_2\bar{p}_1\cdot\bar{p}_2 - A_1\bar{p}_1\cdot\bar{p}_2 - A_2\bar{p}_1^2\right] + (A_1+A_2)v + x\sqrt{y^2-1}\,(v_1+v_2) + x^2(y^2-1)u_3\Big\}\,. \quad (4.81)$$

Using the asymptotic forms of $\bar{p}_1^2$, $\bar{p}_2^2$, and $\bar{p}_1\cdot\bar{p}_2$ given in Eqs.(3.4) and (3.5) yields the result

$$R = \frac{1}{x^2}\Big\{xy\,u + (v - \sqrt{y^2-1}\,u_1)(A_1+A_2)v + x\sqrt{y^2-1}\,(v_1+v_2) + x^2(y^2-1)u_3\Big\} \quad (4.82)$$

The terms involving the functions v and u_1 can be simplified further. Substituting the expressions for these functions found in Eqs.(4.51) and (4.75) gives

$$v - \sqrt{y^2-1}\,u_1 = c_1(y^2-1)^{\frac{1}{2}\mu}\left[P_\nu^\mu(y) - (2\mu+1)\sqrt{y^2-1}\,P_\mu^{\mu-1}(y)\right]\,. \quad (4.83)$$

Using equation (E.46) found in Appendix E, it follows that the previous equation can be written as

$$v - \sqrt{y^2-1}\,u_1 = c_1(y^2-1)^{\frac{1}{2}\mu}\left[P_\nu^\mu(y) + P_{\mu-1}^\mu(y) - P_{\mu+1}^\mu(y)\right]\,. \quad (4.84)$$

If we notice that $\mu+1=\nu$, this expression becomes

$$v - \sqrt{y^2-1}\,u_1 = c_1(y^2-1)^{\frac{1}{2}\mu}P_{-\mu}^\mu(y)\,. \quad (4.85)$$

Furthermore, using Eq.(E.41) it follows that

$$v - \sqrt{y^2-1}\,u_1 = c_1\,2^\mu/\Gamma(1-\mu)\,. \quad (4.86)$$

or, using Eq.(4.80)

$$v - \sqrt{y^2 - 1}\, u_1 = (1 - \frac{1}{2}\eta) \; . \tag{4.87}$$

If we substitute this expression in Eq.(4.83) and set all other functions equal to zero, we arrive at the equation

$$\Box_v^2 u = -3\epsilon(1 - \frac{1}{2}\eta)(A_1 + A_2)/x^2 \; . \tag{4.88}$$

Using the same approach used in solving the first five differential equtions, we assume that the u-function can be represented by the form

$$u = x^{\beta}\bar{u}(y) \; . \tag{4.89}$$

Substituting the previous form into equation (4.88) and using the asymptotic form of the D'alambertian operator as well as the asymptotic forms of the functions A_1 and A_2 gives

$$x^{\beta-1}4(\beta+2)\hat{D}_{\beta}\bar{u} = -3\epsilon(1 - \frac{1}{2}\eta)\, x^{-2-\eta}(z^{\eta} + z^{-\eta}) \; . \tag{4.90}$$

Matching the x-dependency on both sides of Eq.(4.90) implies that the parameter β must be defined by the formula

$$\beta = -(1+\eta) \; . \tag{4.91}$$

Substituting the form of the $\hat{D}$ given in Eq.(4.4) yields the equation

$$(y^2 - 1)\bar{u}\,' + (1+\eta)y\bar{u} = \lambda(z^{\eta} + z^{-\eta}) \; , \tag{4.92}$$

where

$$\lambda = \frac{3}{4}\epsilon\, \frac{(1 - \frac{1}{2}\eta)}{(1-\eta)} \; . \tag{4.93}$$

To solve this equation, we notice that, for values of $y \geq 10$, the approximation $z \sim 2y$ is very good. This leads to a relative error of less than .2%. Since the perturbation solutions are good in this region, we will attempt to solve the approximate equation

$$y^2\bar{u}' + (1+\eta)y\bar{u} = \lambda\left[(2y)^{\eta} + (2y)^{-\eta}\right] \ . \tag{4.94}$$

In order to solve this equation, we make the change of variables

$$\xi = (2y)^{\eta}\,, \quad d\xi = 2\eta\xi^{(1-\frac{1}{\eta})}\ , \quad \text{and} \quad y = \frac{1}{2}\xi^{\frac{1}{\eta}}\ . \tag{4.95}$$

Subtituting these expressions in Eq.(4.94) and simplifying yields the differential equation

$$\bar{u}^* + (1+1/\eta)\,\frac{u}{\xi} = 2(\lambda/\eta)\xi^{-\frac{1}{\eta}}[1+\xi^{-2}]\ , \tag{4.96}$$

where the symbol $\bar{u}^*$ stands for the derivative of $\bar{u}$ with respect to ξ. This equation can be solved by introducing the integrating factor

$$I(\xi) = exp\left\{(1+1/\eta)\int\frac{d\xi}{\xi}\right\} = \xi^{(1+\frac{1}{\eta})}\ . \tag{4.97}$$

The general solution can be written in terms of the integrating factor in the form

$$\bar{u} = \frac{d}{I(\xi)} + (2\lambda/\eta)\frac{1}{I(\xi)}\int^{\xi}\left\{t^{-\frac{1}{\eta}}[1+t^{-2}]\right\}I(t)\,dt\ , \tag{4.98}$$

where d is an arbitrary constant. The integration in the previous equation is elementary an yields the result

$$\bar{u} = d\xi^{-(1+\frac{1}{\eta})} + (2\lambda/\eta)\xi^{-(1+\frac{1}{\eta})}\left[\ln\xi + \frac{1}{2}\xi^2\right]\ . \tag{4.99}$$

In terms of the variable y, the solution can be written as

$$\bar{u} = \frac{1}{2y}\left\{d(2y)^{-\eta} + (\lambda/\eta)(2y)^{\eta} + 2\lambda(2y)^{-\eta}\ln(2y)\right\}\ . \tag{4.100}$$

To determine the constant d, we require that this solution join smoothly with the asymptotic form of the perturbation solution given in Eq.(C.46) of Appendix C.

In order to join the solution just found, we notice that for values y not too large the following expansions are valid due to the small magnitude of the constant η ($\eta \approx 1.75 \times 10^{-3}$):

$$\begin{aligned} (2y)^{\eta} \ &= exp\{\ \ \eta \ln(2y)\} = 1 + \eta \ln(2y) + \cdots \ , \\ (2y)^{-\eta} &= exp\{-\eta \ln(2y)\} = 1 - \eta \ln(2y) + \cdots \ . \end{aligned} \tag{4.101}$$

If we select the value of the constant d to be equal to $-\lambda/\eta$, then Eq.(4.97) becomes

$$\bar{u} = \frac{\lambda}{2y}\left\{\frac{1}{\eta}[(2y)^{\eta} - (2y)^{-\eta}] + 2(2y)^{-\eta}\ln(2y)\right\} \ , \tag{4.102}$$

or because of the expansion given in Eq.(4.101),

$$\bar{u} \rightarrow \frac{2\lambda}{y}\ln(2y)[1 - \eta \ln(2y)] \ . \tag{4.103}$$

Substituting the expression for λ given in Eq.(4.90) gives

$$\bar{u} \rightarrow \frac{3}{2}\epsilon\frac{\ln(2y)}{y} \ + \ \mathcal{O}(\epsilon^2) \ , \tag{4.104}$$

which has the same form as that given in Eq.(C.46). Therefore, the solution is found to be given by the expression

$$\bar{u} = \frac{1}{2y}(1 - \tfrac{3}{8}\epsilon)\left\{(2y)^{\eta} - (2y)^{-\eta} + 2\eta(2y)^{-\eta}\ln(2y)\right\} \ . \tag{4.105}$$

It is not difficult to construct a "solution" that is accurate near $y = 1$. From the form of the perturbation solution near $y = 1$ and the fact that we can replace all terms involving z by $(2y)$, it is reasonable to assume that this replacement will improve the solution in this region. With this argument the improved solution is given by the expression

$$\bar{u} = \frac{1}{2}(1 - \tfrac{3}{8}\epsilon)\left\{(z^{\eta} - z^{-\eta}) + 2\eta\, z^{-\eta}\ln z\right\}/\sqrt{y^2 - 1} \ . \tag{4.106}$$

For reference purpose, the asymptotic solutions are presented in the Table 4-1. Numerical values for the functions $\bar{v}$, $\bar{e}$, $\bar{u}_1$, $\bar{v}_3$ and $\bar{u}$ are presented in Table 4-2. The variable x inside the overlapping region was chosen to be 10^{20}. Their derivatives are also presented. The values of the perturbations solutions are also shown for comparison.

Table 4-1 Asymptotic Solutions to the Model Equation

Notation:

$$\mu = \tfrac{1}{2}(-1 + \tfrac{1}{2}\epsilon); \qquad \nu = \tfrac{1}{2}(1 + \tfrac{1}{2}\epsilon); \qquad \eta = \frac{3}{4}\epsilon .$$

$$v = (1 - \frac{1}{2}\eta)(y^2 - 1)^{\frac{1}{2}\mu} P_\nu^\mu(y) . \tag{4.107}$$

$$e = -(1 - \frac{1}{2}\eta)(y^2 - 1)^{\frac{1}{2}\mu} P_\mu^\mu(y)/x \tag{4.108}$$

$$u_1 = -u_2 = (1 - \frac{1}{2}\eta)(2\mu + 1)(y^2 - 1)^{\frac{1}{2}\mu} P_\mu^{\mu-1}(y)/x . \tag{4.109}$$

$$v_3 = -\tfrac{1}{2}\epsilon(1 - \frac{1}{2}\eta)(y^2 - 1)^{\frac{1}{2}\mu} P_\mu^\mu(y)/x . \tag{4.110}$$

$$u = \frac{1}{2}(1 - \tfrac{3}{8}\epsilon)x^{-(1+\eta)} \left\{(z^\eta - z^{-\eta}) + 2\eta\, z^{-\eta} \ln z\right\} / \sqrt{y^2 - 1} . \tag{4.111}$$

Table 4-2 Perturbation and Asymptotic Solutions in the Overlapping Region

Perturbation Solutions ($x = 10^{20}$ and $y = 1.1$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	9.957E-01	-9.679E-01	1.640E-04	1.124E-03	3.372E-03
1^{st} *derivative*	7.301E-04	3.083E-01	7.341E-04	-3.579E-04	-1.074E-03
2^{nd} *derivative*	-4.155E-04	-2.352E-01	-4.877E-03	2.731E-04	8.193E-04

Asymptotic Solutions ($x = 10^{20}$ and $y = 1.1$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	9.992E-01	-9.671E-01	1.640E-04	1.123E-03	3.373E-03
1^{st} *derivative*	7.298E-04	3.078E-01	7.338E-04	-3.574E-04	-1.081E-03
2^{nd} *derivative*	-4.149E-04	-2.348E-01	-4.875E-03	2.726E-04	8.574E-04

Perturbation Solutions ($x = 10^{20}$ and $y = 10$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	9.979E-01	-3.008E-01	2.344E-04	3.493E-04	1.048E-03
1^{st} *derivative*	1.138E-04	2.029E-02	-1.224E-05	-2.355E-05	-7.066E-05
2^{nd} *derivative*	-1.101E-05	-3.109E-03	1.390E-06	3.609E-06	1.083E-05

Asymptotic Solutions ($x = 10^{20}$ and $y = 10$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	1.001E+00	-3.008E-01	2.345E-04	3.493E-04	1.046E-03
1^{st} *derivative*	1.139E-04	2.027E-02	-1.223E-05	-2.354E-05	-7.063E-05
2^{nd} *derivative*	-1.102E-05	-3.105E-03	1.388E-06	3.606E-06	1.083E-05

Perturbation Solutions ($x = 10^{20}$ and $y = 100$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	1.001E+00	-5.299E-02	4.991E-05	6.152E-05	1.846E-04
1^{st} *derivative*	1.161E-05	4.299E-04	-3.831E-07	-4.992E-07	-1.497E-06
2^{nd} *derivative*	-1.160E-07	-7.599E-06	6.504E-09	8.823E-09	2.647E-08

Asymptotic Solutions ($x = 10^{20}$ and $y = 100$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	1.004E+00	-5.305E-02	5.000E-05	6.160E-05	1.839E-04
1^{st} *derivative*	1.165E-05	4.301E-04	-3.835E-07	-4.994E-07	-1.493E-06
2^{nd} *derivative*	-1.163E-07	-7.601E-06	6.508E-09	8.825E-09	2.641E-08

Table 4-1 (Continued)

Perturbation Solutions ($x = 10^{20}$ and $y = 1000$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	1.003E+00	-7.601E-03	7.664E-06	8.825E-06	2.648E-05
1^{st} *derivative*	1.161E-06	6.601E-06	-6.503E-09	-7.664E-09	-2.299E-08
2^{nd} *derivative*	-1.161E-09	-1.220E-08	1.185E-11	1.417E-11	4.251E-11

Asymptotic Solutions ($x = 10^{20}$ and $y = 1000$)

	$\bar{v}$	$\bar{e}$	$\bar{u}_1$	$\bar{v}_3$	$\bar{u}$
	1.007E+00	-7.620E-03	7.687E-06	8.848E-06	2.633E-05
1^{st} *derivative*	1.169E-06	6.613E-06	-6.518E-09	-7.679E-09	-2.288E-08
2^{nd} *derivative*	-1.168E-09	-1.222E-08	1.187E-11	1.419E-11	4.232E-11

CHAPTER V

THE CONCLUSION

A solution for the vertex amplitude has been found to the Schwinger-Dyson equations based on an approximation scheme which is characterized by the following:

(1) the photon propagator is approximated by its form near the mass shell,

(2) the infinite hierarchy of the vertex is cut off at the second order in the coupling constant and the remainder is approximated by Green's generalization of the Ward Identity for higher order contributions.

The gauge is chosen to be that found necessary to obtain a finite solution to the electron propagator equation[20] with a vanishing bare mass for the electron. In the above reference, the electron propagator is found to be given by

$$\underline{S}^{-1}(\bar{p}) \cong -A(\bar{p}^2) + B(\bar{p}^2)\not{\bar{p}} \tag{5.1}$$

where

$$A(\bar{p}^2) = m\left|\frac{\bar{p}^2}{m^2} - 1\right|^{3\alpha(m^2-\bar{p}^2)/4\pi\bar{p}^2}, \tag{5.2}$$

$$B(\bar{p}^2) \cong 1 \ . \tag{5.3}$$

In the asymptotic region, a simpler form of the A-function was used, namely

$$A(\bar{p}^2) \sim (\bar{p}^2/m^2)^{-\eta} \quad \eta \approx 1.75 \times 10^{-3} \ . \tag{5.4}$$

A simplified approximation of the vertex equation (the model equation) was found. This tensor and matrix equation can be reduced to a set of 8 coupled

differential equations of third order in two variables. The order of the equations was reduced by the introduction of an additional function e. The set of 8 *basic* scalar functions is needed to define the vertex completely as required by the transformation properties of the vertex amplitude. In general, the transverse part of the vertex amplitude can be written in the form

$$\begin{aligned}
\tilde{\underline{\Gamma}}^{\lambda}(\bar{p}_1,\bar{p}_2) = {} & \tilde{\underline{\gamma}}^{\lambda}\, v(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + \tilde{p}^{\lambda} u(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + \not{p}_1 \tilde{p}^{\lambda} u_1(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + \not{p}_2 \tilde{p}^{\lambda} u_2(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + \tilde{p}^{\lambda}[\not{p}_1,\not{p}_2] u_3(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + [\not{p}_1,\tilde{\underline{\gamma}}^{\lambda}] v_1(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + [\not{p}_2,\tilde{\underline{\gamma}}^{\lambda}] v_2(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2) \\
& + i\varepsilon^{\lambda\mu\nu\rho}\underline{\gamma}_{\mu} p_{1\nu} p_{2\rho} v_3(\bar{p}_1^2,\bar{p}_2^2,\bar{p}_1\cdot\bar{p}_2)\ .
\end{aligned} \tag{5.5}$$

The longitudinal part of the vertex is given by Ward's identity (see Eq.(2.27)).

The analytical approximate solution to this simpler equation has been found in the vicinity of the four momenta square equal to the square of the experimental mass.[23,24] The validity of these solutions was tested by direct substitution into the differential equations (see Eqs.(3.71) to (3.79) with Eqs.(2.85) to (2.92)). A measurement of their accuracy is given by the difference between the left and right-hand side of the differential equations divided by their average. These solutions were found to be valid within .5% for 6 of the equations (v, u_1, u_2, v_3, e, and u) and a much higher error (3%) for the remaining functions (v_1, v_2, u_3). The region of validity of these solutions was found to be $x > 1$ and $1 < y < 10^4$.

The introduction of the variables,

$$x = \sqrt{\bar{p}_1^2 \bar{p}_2^2} \qquad \text{and} \qquad z = \sqrt{\bar{p}_1^2 / \bar{p}_2^2}\,, \tag{5.6}$$

made it possible to separate the system of 9 coupled differential equations into three *separate* systems of five, three and one differential equations in the asymptotic region of large momenta. Using these variables, the solutions factor into functions of x and functions of z in this region, and the x-dependency of all the solutions can be written in a simple power law form. This reduces the problem to finding functions of only one variable.

The goal of this work was to find the solutions to the set of differential equations containing the largest component of the vertex amplitude, namely the v-function. The differential equations for the functions v, u_1, u_2, v_3, e, are linked together but do not involve the other dependent variables (see Eqs.(3.71) to (3.75)). Solutions were found for these five functions that join smoothly with the asymptotic forms of the perturbation solutions. The percent difference between the perturbation and asymptotic solutions is less than .1% in the region $x/k^2 > 10^{10}$ and $1.1 \geq y \geq 10^6$ with a similar result for their first and second derivatives. This confirms the results[24] already found that the perturbation solutions gives a good representation of the vertex amplitude in a very large region of the electron momentum. This conclusion follows from the fact that the largest part of the vertex amplitude (*i.e.* $v\underline{\tilde{\gamma}}^\lambda$) probably has a maximum at the mass shell. The smallness of the fine structure constant and the slowness of the decay of the electron propagator function A in the asymptotic region are also responsible for this result.

These results were encouraging so that an attempt was made in solving the u-function. The techniques and general insight gained in the solution of the v-function proved useful. This attempt was successful and a solution was found that also joined smoothly to the perturbation.

The form of these functions, except the v-function, vanishes at infinity as rapidly as $\mathcal{O}(1/x)$ and therefore is negligible compared to the v-function. Also, these functions are one order in the coupling parameter smaller than the v-function. We are lead to conclude that the most important function in defining the vertex amplitude is v. Although the perturbation solution gives a good representation of this function in the intermediate region, this form may *not* be useful in determining the convergence of several interesting quantities. The analytical properties of the vacuum polarization integral and charge renormalization constant Z_3 are strongly dependent of the analytical properties of the vertex amplitude. The convergence of this integral may well depend on the way the vertex amplitude behaves at infinity. The perturbation expression given in equation $(C.41)$ shows a logarithmically divergent form for this function as the variable z growth without bounds. The present work has demonstrated that the behavior is much less rapid ($v \sim z^{\frac{1}{2}\epsilon}$).

This result alone encourages the calculation of the vertex amplitude to higher accuracy to determine if it is constant everywhere or even decreases for large values of the variables. To address this question, a recalculation of the photon propagator is needed. The form of the solutions found in this work may lead to the calculation of a correction to the photon propagator. This in turn will provide a way to calculate the renormalization constant Z_3 and the bare charge of the electron.

It cannot be said that the present solutions, described in this work, constitute a complete resolution of the problems in the theory of QED. However, this method has been successful in finding these solutions without the unreasonable infinite quantities that plague renormalization theory. Its felt that within the level of approximations made, the present solutions give evidence that the infinite quantities that occur in the usual perturbation calculations of the self-energy of the electron and the vertex amplitude are not essential to the theory. Furthermore, the present

work sets the foundations for the subsequent calculation of the photon propagator and the closing of the iteration procedure of the project.

APPENDIX A

DIRAC MATRICES: DEFINITION AND IDENTITIES

The relativistically invariant equations which in the case of the electron play the same role as the Maxwell's equations do in the case of the photon, were obtain by P.A.M. Dirac[30] in 1927. In order to describe these equations in a relativistically invariant form, Dirac introduced a set of four-dimensional matrices (the Dirac matrices) given by

$$\underline{\gamma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \;;\; \underline{\gamma}^j = \begin{pmatrix} 0 & \underline{\sigma}^j \\ -\underline{\sigma}^j & 0 \end{pmatrix} \;;\; j = 1, 2, 3 \tag{A.1}$$

where $\underline{\sigma}^j$ represents the Pauli spin matrices. The Pauli spin matrices are two-dimensional matrices defined by

$$\underline{\sigma}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;;\; \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;;\; \underline{\sigma}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} . \tag{A.2}$$

The Dirac gamma matrices satisfy the anticommutator relation,

$$\underline{\gamma}_\mu \underline{\gamma}_\nu + \underline{\gamma}_\nu \underline{\gamma}_\mu = 2g_{\mu\nu} . \tag{A.3}$$

The multiple products of gamma matrices form a group of 16 linearly independent 4×4 matrices. One possible representation of this group is given by the linear combination of products defined by the matrices $I, \underline{\gamma}_\mu, \underline{\gamma}_{\mu\nu}, \underline{\gamma}_{\lambda\mu\nu}$ where

$$\underline{\gamma}_{\mu\nu} = \frac{1}{2}[\underline{\gamma}_\mu, \underline{\gamma}_\nu] \tag{A.4}$$

and

$$\underline{\gamma}_{\mu\nu\rho} = \frac{1}{2}\{\underline{\gamma}_\mu, \underline{\gamma}_{\nu\rho}\} , \tag{A.5}$$

and I is the 4-dimensional unit matrix. We have used the symbols [,] and {,} to represent the commutator and anticommutator respectively. Another useful matrix, used in defining the model equation is defined as

$$\underline{\gamma}^5 = -i\underline{\gamma}^0\underline{\gamma}^1\underline{\gamma}^2\underline{\gamma}^3 . \tag{A.6}$$

Equation $(A.6)$ may be used to write Eq.$(A.5)$ in the form

$$\underline{\gamma}_{\mu\nu\rho} = i\varepsilon_{\mu\nu\rho\alpha}\underline{\gamma}_5\underline{\gamma}^\alpha . \tag{A.7}$$

The symbol $\varepsilon_{\mu\nu\rho\alpha}$ stands for the 4-rank antisymmetric tensor (Levi Civita) defined by the rules

$$\varepsilon_{\mu\nu\rho\alpha} = 0 \qquad \text{unless } \mu, \nu, \rho, \alpha \text{ are all different} , \tag{A.8}$$

$$\varepsilon_{1234} = 1 , \tag{A.9}$$

$$\varepsilon^{\mu\nu\rho\sigma} = \begin{cases} +1 & \text{for even permutation of the indices 1,2,3,4;} \\ -1 & \text{for odd permutation of the indices 1,2,3,4;} \end{cases} \tag{A.10}$$

and

$$\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} -1 & \text{for even permutation of the indices 1,2,3,4;} \\ +1 & \text{for odd permutation of the indices 1,2,3,4.} \end{cases} \tag{A.11}$$

It is simple to verify that the gamma matrices satisfy also the following identities:

$$\underline{\gamma}^\lambda\underline{\gamma}_\lambda = 4 , \tag{A.12}$$

$$\underline{\gamma}^\lambda\underline{\gamma}_\mu\underline{\gamma}_\lambda = -2\underline{\gamma}_\mu , \tag{A.13}$$

$$\underline{\gamma}^\lambda\underline{\gamma}_{\mu\nu}\underline{\gamma}_\lambda = 0 , \tag{A.14}$$

$$\underline{\gamma}^\lambda\underline{\gamma}_{\mu\nu\rho}\underline{\gamma}_\lambda = 2\underline{\gamma}_{\mu\nu\rho} . \tag{A.15}$$

By definition, the operator $\not{\nabla} = \underline{\gamma}^\mu \partial_\mu$ and therefore, using Eqs.($A.12$) to ($A.15$) it is possible to show that

$$\not{\nabla}\underline{\gamma}_\mu = -\underline{\gamma}_\mu \not{\nabla} + 2\partial_\mu \ , \tag{A.16}$$

$$\not{\nabla}\underline{\gamma}_{\mu\nu} = \underline{\gamma}_{\mu\nu}\not{\nabla} + 2(\partial_\mu \underline{\gamma}_\nu - \partial_\nu \underline{\gamma}_\mu) \ , \tag{A.17}$$

$$\not{\nabla}\underline{\gamma}_{\mu\nu\rho} = -\underline{\gamma}_{\mu\nu\rho}\not{\nabla} + 2(\partial_\mu \underline{\gamma}_{\nu\rho} + \partial_\nu \underline{\gamma}_{\rho\mu} + \partial_\rho \underline{\gamma}_{\mu\nu}) \ . \tag{A.18}$$

APPENDIX B

DERIVATION OF GREEN'S PERTURBATION SOLUTIONS

To obtain the perturbation solutions to the model equation, one sets $C_\mu^\lambda \sim \tilde{\delta}_\mu^\lambda$,$C^\lambda \sim C_{\mu\nu}^\lambda \sim C_{\mu\nu\rho}^\lambda \sim 0$ and $A_1 \sim A_2 \sim m$ as the zeroth order approximation in equations (2.52) to (2.55). Therefore, the D tensors become

$$D^\lambda = \quad 2m\tilde{p}^\lambda/D \ , \tag{B.1}$$

$$D_\mu^\lambda = \quad \left[(m^2 - \bar{p}_1 \cdot \bar{p}_2)\tilde{\delta}_\mu^\lambda + (p_{1\mu} + p_{2\mu})\tilde{p}^\lambda\right]/D \ , \tag{B.2}$$

$$D_{\mu\nu}^\lambda = \quad m\left[(m^2 - \bar{p}_1 \cdot \bar{p}_2)\tilde{\delta}_\mu^\lambda - (p_{1\mu} - p_{2\mu})\tilde{\delta}_\mu^\lambda\right]/D \ , \tag{B.3}$$

$$\begin{aligned} D_{\mu\nu\rho}^\lambda = \quad & m\left[(p_{1\mu}p_{2\nu} - p_{1\nu}p_{2\mu})\tilde{\delta}_\rho^\lambda + (p_{1\nu}p_{2\rho} - p_{1\rho}p_{2\nu})\tilde{\delta}_\mu^\lambda\right. \\ & \left. + (p_{1\rho}p_{2\mu} - p_{1\mu}p_{2\rho})\tilde{\delta}_\nu^\lambda\right]/D \ , \end{aligned} \tag{B.4}$$

where $\bar{p}_1 = \bar{p} + \bar{k}$ and $\bar{p}_2 = \bar{p} - \bar{k}$. To solve the model equation under these approximations one must substitute equations $(B.1)$ to $(B.4)$ into equations (2.66) through (2.69) and integrate. This can be accomplished using the following functional relation. Let $X_q = (\bar{x} + \bar{q})^2 = (\bar{x}^2 + 2\bar{x} \cdot \bar{q} + \bar{q}^2)$. Then it follows that

$$\Box^2\left[x_q^{-1}F(x_q)\right] = 4F'' \equiv G(x_q) \tag{B.5}$$

and therefore,

$$F(x_q) = \frac{1}{4}\int_0^{x_q} dx_q' \int_0^{x_q'} dx_q'' G(x_q'') \ . \tag{B.6}$$

Using this relation, one can easily show that, if

$$\Box^2\Phi(x_q/m^2) = \frac{1}{x_q - m^2} \ , \tag{B.7}$$

then

$$\Phi(x_q/m^2) = \frac{1}{4}(1 - \frac{m^2}{x_q}) \ln(\frac{x_q}{m^2} - 1) \ . \tag{B.8}$$

Also, if one defines

$$X_\xi = (\bar{p} + \xi\bar{k})^2, \quad \mu_\xi = (m^2 - \bar{k}^2) + \bar{k}^2\xi^2 \quad \text{and} \quad Z_\xi = X_\xi/\mu_\xi \tag{B.9}$$

then

$$\frac{1}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} = \frac{1}{2}\int_{-1}^{1} \frac{d\xi}{\mu_\xi(Z_\xi - 1)} \ . \tag{B.10}$$

With this definition it follows directly that if,

$$\Box^2\Phi_a = \frac{1}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \tag{B.11}$$

then

$$\Phi_a = -\frac{1}{8}\int_{-1}^{1} \frac{d\xi}{X_\xi} \ln(Z_\xi - 1) \ . \tag{B.12}$$

It is easily shown that one can write

$$\frac{\ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right)}{\bar{p}_1^2 - \bar{p}_2^2} = \frac{1}{2}\int_{-1}^{1} \frac{d\xi}{X_\xi - \mu_\xi} \tag{B.13}$$

so that, if

$$\Box^2\Phi_b = \frac{\ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right)}{\bar{p}_1^2 - \bar{p}_2^2} \ , \tag{B.14}$$

then

$$\Phi_b = \frac{1}{2}\int_{-1}^{1} d\xi \Phi(Z_\xi) = \frac{1}{8}\int_{-1}^{1} d\xi (1 - \frac{1}{Z_\xi}) \ln(Z_\xi - 1) \ . \tag{B.15}$$

Similarly if

$$\Box^2\Phi_c = \int_{-1}^{1} \frac{d\xi}{X_\xi}(m^2 - \mu_\xi) \ln(Z_\xi - 1) \ , \tag{B.16}$$

then

$$\Phi_c = \frac{1}{4}\int_{-1}^{1} d\xi \left\{ L_2(Z_\xi) - (1 - \frac{1}{z_\xi}) \ln(Z_\xi - 1) \right\} \tag{B.17}$$

where $L_2 = -\int_0^z \ln(1-z')\frac{dz'}{z'}$ is the dilogarithm. The dilogarithm satisfies the integral relation

$$\int_0^z L_2(z')dz' = zL_2(z) + (z-1)\ln(1-z) - z \ . \tag{B.18}$$

With these relations, it is possible to solve equations (2.66) to (2.69) as follows: Equation (2.66) with (B.1) on the right-hand side becomes

$$\Box^2 C^\lambda = -\frac{6\epsilon m\tilde{p}^\lambda}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} = 3\epsilon m \frac{\partial}{\partial \tilde{p}_\lambda} \frac{\ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right)}{\bar{p}_1^2 - \bar{p}_2^2} \ . \tag{B.19}$$

Integrating Eq.(B.19) using Eq.(B.15) gives

$$C^\lambda = \frac{3}{4}\epsilon m\tilde{p}^\lambda \int_{-1}^1 \frac{d\xi}{\mu_\xi Z_\xi}\left\{1 + \frac{\ln(Z_\xi - 1)}{Z_\xi}\right\} \ . \tag{B.20}$$

Next, substituting Eq.(B.2) into Eq.(2.67) yields

$$\Box^2 C_\mu^\lambda = 2\epsilon\left\{\frac{(m^2 - \bar{p}_1\cdot\bar{p}_2)\tilde{\delta}_\mu^\lambda + (p_{1\mu} + p_{2\mu})\tilde{p}^\lambda}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} - \frac{\partial E^\lambda}{\partial p_\mu}\right\} \tag{B.21}$$

where E^λ satisfies the differential equation

$$\Box^2 E^\lambda = \frac{\partial}{\partial p_\nu}\left\{\frac{(m^2 - \bar{p}_1\cdot\bar{p}_2)\tilde{\delta}_\nu^\lambda + (p_{1\nu} + p_{2\nu})\tilde{p}^\lambda}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)}\right\} \ . \tag{B.22}$$

One can easily verify that the following relation holds true:

$$\frac{(m^2 - \bar{p}_1\cdot\bar{p}_2)\tilde{\delta}_\mu^\lambda + (p_{1\mu} + p_{2\mu})\tilde{p}^\lambda}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} = -\frac{1}{2}\frac{\partial}{\partial\tilde{p}_\lambda}\left\{\frac{p_{1\mu} + p_{2\mu}}{(\bar{p}_1^2 - \bar{p}_2^2)}\ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right)\right\}$$

$$+\tilde{\delta}_\mu^\lambda\left\{\frac{(m^2 - \bar{p}_1\cdot\bar{p}_2)}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} + \frac{1}{\bar{p}_1^2 - \bar{p}_2^2}\ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right)\right\} \ . \tag{B.23}$$

Therefore, C_μ^λ can be written in the form

$$C_\mu^\lambda = C\tilde{\delta}^\lambda + \frac{\partial}{\partial\tilde{p}_\lambda}C_\mu \ , \tag{B.24}$$

where $\mathcal{C}$ and $\mathcal{C}_\mu$ satisfy the differential equations

$$\Box^2 \mathcal{C}_\mu = 2\epsilon \left\{ \frac{1}{2} \frac{p_{1\mu} + p_{2\mu}}{(\bar{p}_1^2 - \bar{p}_2^2)} \ln \left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) + \frac{\partial E_1}{\partial p_\mu} \right\} , \qquad (B.25)$$

$$\begin{aligned} \Box^2 \mathcal{C} &= 2\epsilon \left\{ \frac{(m^2 - \bar{p}_1 \cdot \bar{p}_2)}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} + \frac{1}{\bar{p}_1^2 - \bar{p}_2^2} \ln \left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) \right\} \\ &= \epsilon \Bigg\{ - \frac{1}{(\bar{p}_1^2 - m^2)} - \frac{1}{(\bar{p}_2^2 - m^2)} + \frac{4\bar{k}^2}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \\ &\quad + \frac{2}{\bar{p}_1^2 - \bar{p}_2^2} \ln \left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) \Bigg\} , \end{aligned} \qquad (B.26)$$

and

$$\Box^2 E_1 = \frac{2m^2}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} - \frac{2}{\bar{p}_1^2 - \bar{p}_2^2} \ln \left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) . \qquad (B.27)$$

Using Eqs. $(B.25)$ to $(B.27)$ it is found that

$$\Box^2 \left(\mathcal{C} - \frac{\partial \mathcal{C}_\mu}{\partial p^\mu} \right) = 0 \qquad (B.28)$$

which implies that $\mathcal{C} = \frac{\partial \mathcal{C}_\mu}{\partial p^\mu}$. This also imply that

$$\partial^\mu C_\mu^\lambda = 0 \, . \qquad (B.29)$$

This relation is valid in general for the model equation as can be seen from a direct application of ∂^μ to Eq.(2.67). From equations $(B.8)$ and $(B.12)$, one obtains

$$\mathcal{C} = -\epsilon \left\{ \Phi \left(\frac{\bar{p}_1^2}{m^2} \right) + \Phi \left(\frac{\bar{p}_2^2}{m^2} \right) \right\} + 4\epsilon \bar{k}^2 \Phi_a + 2\epsilon \Phi_b$$

or

$$\begin{aligned} \mathcal{C} = \frac{1}{4}\epsilon \Bigg\{ &\int_{-1}^{1} \left(1 - \frac{1}{Z_\xi} - \frac{2\bar{k}^2}{\mu_\xi Z_\xi} \right) \ln(Z_\xi - 1) d\xi \\ &- \left(1 - \frac{m^2}{\bar{p}_1^2} \right) \ln \left(\frac{\bar{p}_1^2}{m^2} - 1 \right) - \left(1 - \frac{m^2}{\bar{p}_2^2} \right) \ln \left(\frac{\bar{p}_2^2}{m^2} - 1 \right) \Bigg\} , \end{aligned} \qquad (B.30)$$

and from equation $(B.16)$ with $\tilde{\mathcal{C}}^\mu = \mathcal{C}^\mu - k^\alpha \mathcal{C}_\alpha k^\mu/k^2$, one finds

$$\begin{aligned}
\Box^2 \tilde{\mathcal{C}}_\mu &= 2\epsilon \left\{ \frac{\tilde{p}_\mu}{(\bar{p}_1^2 - \bar{p}_2^2)} \ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2}\right) + \frac{\partial E_1}{\partial \tilde{p}^\mu} \right\} \\
&= 2\epsilon \frac{\partial}{\partial \tilde{p}^\mu} \left\{ \frac{1}{4} \int_{-1}^{1} \ln(Z_\xi - 1) d\xi + E_1 \right\} \\
&= 2\epsilon \frac{\partial}{\partial \tilde{p}^\mu} \left\{ \frac{1}{4} \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} (m^2 - \mu_\xi) \ln(Z_\xi - 1) \right\} . \qquad (B.31)
\end{aligned}$$

This follows from equations $(B.12)$,$(B.15)$ and $(B.22)$ from which

$$E_1 = 2m^2 \Phi_a - 2\Phi_b = -\frac{1}{4} \int_{-1}^{1} \frac{d\xi}{Z_\xi} \left(Z_\xi + \frac{m^2}{Z_\xi} - 1 \right) \ln(Z_\xi - 1) + constant \, . \quad (B.32)$$

If we define the function e such that $E^\lambda = e\tilde{p}^\lambda$, then it follows that $e = 2\partial E_1/\partial X_\xi$ and thus

$$e = -\frac{1}{2} \int_{-1}^{1} d\xi \left\{ \frac{1}{\mu_\xi (Z_\xi - 1)} + \frac{(m^2 - \mu_\xi)}{\mu_\xi^2} \left[\frac{1}{Z_\xi (Z_\xi - 1)} - \frac{\ln(Z_\xi - 1)}{Z_\xi^2} \right] \right\} . (B.33)$$

Using Eq.$(B.16)$ one can calculate

$$\begin{aligned}
\tilde{\mathcal{C}}_\mu &= -\frac{1}{2} \epsilon \frac{\partial}{\partial \tilde{p}^\mu} \int_{-1}^{1} d\xi (m^2 - \mu_\xi) \{ L_2(Z_\xi) + (1 - \frac{1}{Z_\xi}) \ln(Z_\xi - 1) \} \\
&= \frac{1}{4} \tilde{p}_\mu \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} (m^2 - \mu_\xi) \left\{ 1 - \left(1 - \frac{1}{Z_\xi} \right) \ln(Z_\xi - 1) \right\} . \qquad (B.34)
\end{aligned}$$

It also follows from Eqs. $(B.16)$ and $(B.18)$ that

$$\Box^2 k^\mu \mathcal{C}_\mu = 2\epsilon \left\{ \frac{1}{4} \ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) + \bar{k} \cdot \tilde{\nabla} E_1 \right\} \qquad (B.35)$$

and

$$\Box^2 (k^\mu \mathcal{C}_\mu - \epsilon \bar{p} \cdot \bar{k} E_1) = \epsilon \ln\left(\frac{\bar{p}_1^2 - m^2}{\bar{p}_2^2 - m^2} \right) - \frac{1}{4} \frac{m^2 (\bar{p}_1^2 - \bar{p}_2^2)}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \, . \qquad (B.36)$$

Therefore

$$k^\mu \mathcal{C}_\mu = -\frac{1}{4} \epsilon \bar{p} \cdot \bar{k} \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} (m^2 - \mu_\xi) \ln(Z_\xi - 1) \qquad (B.37)$$

Using Eq.$(B.34)$ with $(B.37)$ gives

$$\mathcal{C}_\mu = \frac{1}{4}\epsilon\tilde{p}_\mu \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}(m^2-\mu_\xi)\left\{1-\left(1-\frac{1}{Z_\xi}\right)\ln(Z_\xi-1)\right\}$$
$$-\frac{1}{4}\epsilon\left(\frac{\bar{p}\cdot\bar{k}}{k^2}\right)k_\mu \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}(m^2-\mu_\xi)\ln(Z_\xi-1)\ . \tag{B.38}$$

Using Eqs. $(B.24)$ and $(B.38)$ and the fact that the function v multiplies $\tilde{\delta}^\lambda_\mu$ in the definition of C^λ_μ, we finally find that

$$v = \mathcal{C} - \frac{1}{4}\int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}(m^2-\mu_\xi)\left\{1-\left(1-\frac{1}{Z_\xi}\right)\ln(Z_\xi-1)\right\}. \tag{B.39}$$

It also follows from Eqs.$(B.38)$ and (2.77) that if

$$C_k\tilde{p}^\lambda = k^\mu C^\lambda_\mu = \bar{k}\cdot(u_2\bar{p}_1+u_1\bar{p}_2)\tilde{p}^\lambda\ , \tag{B.40}$$

then

$$C_k = \frac{1}{2}\epsilon\bar{p}\cdot\bar{k}\int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}(m^2-\mu_\xi)\left\{\frac{1}{\mu_\xi(Z_\xi-1)}-\frac{1}{\mu_\xi Z_\xi}\ln(Z_\xi-1)\right\}. \tag{B.41}$$

The corresponding equation for C_p is

$$C_p\tilde{p}^\lambda = p^\mu C^\lambda_\mu = [v+\bar{p}\cdot(u_2\bar{p}_1+u_1\bar{p}_2)]\,\tilde{p}^\lambda \tag{B.42}$$

and thus

$$C_p = \frac{1}{2}\epsilon\int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}\left\{\frac{(m^2-\mu_\xi)(m^2-\bar{k}^2)}{\mu_\xi}\left[1+\frac{\ln(Z_\xi-1)}{Z_\xi}\right]-\bar{k}^2\ln(Z_\xi-1)\right\}\ . \tag{B.43}$$

Using Eqs. $(B.41)$ and $(B.43)$ it is possible to solve for the functions u_1 and u_2 simultaneously leading to the expressions

$$u_1 = \frac{(C_p-v)\bar{p}_1\cdot\bar{k}-C_k\bar{p}_1\cdot\bar{p}}{2\left[\bar{p}^2\bar{k}^2-(\bar{p}\cdot\bar{k})^2\right]} \tag{B.44}$$

and

$$u_2 = \frac{C_k\bar{p}_2\cdot\bar{p}-(C_p-v)\bar{p}_2\cdot\bar{k}}{2\left[\bar{p}^2\bar{k}^2-(\bar{p}\cdot\bar{k})^2\right]}\ , \tag{B.45}$$

or, in terms of the ξ-integral transform,

$$u_1 = \frac{1}{4}\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\Big\{2+\Big[\frac{2}{Z_\xi}-1\Big]\ln(Z_\xi-1)\Big\}$$
$$+\frac{1}{4}\epsilon\frac{\bar{p}\cdot\bar{k}}{\bar{k}^2}\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\Big\{\frac{(Z_\xi-2)}{(Z_\xi-1)}+2\frac{\ln(Z_\xi-1)}{Z_\xi}\Big\} \qquad (B.46)$$

and

$$u_2 = \frac{1}{4}\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\Big\{2+\Big[\frac{2}{Z_\xi}-1\Big]\ln(Z_\xi-1)\Big\}$$
$$-\frac{1}{4}\epsilon\frac{\bar{p}\cdot\bar{k}}{\bar{k}^2}\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\Big\{\frac{(Z_\xi-2)}{(Z_\xi-1)}+2\frac{\ln(Z_\xi-1)}{Z_\xi}\Big\}\,. \qquad (B.47)$$

Let us now solve for the functions v_1,v_2 and u_3. To do this, we substitute Eq.(B.3) into Eq.(2.68) which gives

$$\Box^2 C^\lambda_{\mu\nu} = 2\partial_\nu\partial^\rho C^\lambda_{\nu\rho} - 2\partial_\mu\partial^\rho C^\lambda_{\mu\rho} - 2m\epsilon(k_\nu\tilde{\delta}^\lambda_\mu - k_\mu\tilde{\delta}^\lambda_\nu)\frac{1}{(\bar{p}_1^2-m^2)(\bar{p}_2^2-m^2)} \qquad (B.48)$$

where we have used the relation,

$$\Box^2 C^\lambda_{\mu\nu} = \epsilon\tilde{\partial}^\nu D^\lambda_{\mu\nu} = 2m\epsilon(\tilde{\delta}^\lambda_\mu\bar{k}\cdot\Box - k_\mu\tilde{\delta}^\lambda_\mu)\frac{1}{(\bar{p}_1^2-m^2)(\bar{p}_2^2-m^2)}\,. \qquad (B.49)$$

From Eqs.(B.12) and (B.49) it follows that

$$\partial^\nu C^\lambda_{\mu\nu} = 2m\epsilon(\tilde{\delta}^\lambda_\mu\bar{k}\cdot\Box - k_\mu\tilde{\partial}^\lambda)\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi}\ln(Z_\xi-1) \qquad (B.50)$$

and hence, we can define the vector functions

$$F^\lambda_k = k^\mu\partial^\nu C^\lambda_{\mu\nu} = \frac{1}{2}m\epsilon\bar{k}^2\tilde{p}^\lambda\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi}\Big[\frac{1}{(Z_\xi-1)}-\frac{1}{Z_\xi}\ln(Z_\xi-1)\Big] \qquad (B.51)$$

and

$$F^\lambda_p = p^\mu\partial^\nu C^\lambda_{\mu\nu} = -\frac{1}{2}m\epsilon\bar{k}^2\tilde{p}^\lambda\int_{-1}^{1}\frac{\xi d\xi}{\mu_\xi Z_\xi}\Big[\frac{1}{(Z_\xi-1)}-\frac{1}{Z_\xi}\ln(Z_\xi-1)\Big]\,. \qquad (B.52)$$

From Eqs.(B.48) and (B.50), we find that

$$\Box^2 C^\lambda_{\mu\nu} = \frac{1}{2} m\epsilon \left[(\tilde{\delta}^\lambda_\nu \bar{k}\cdot\Box - k_\nu \tilde{\partial}^\lambda)\partial_\mu - (\tilde{\delta}^\lambda_\mu \bar{k}\cdot\Box - k_\mu \tilde{\partial}^\lambda)\partial_\nu \right] \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} \ln(Z_\xi - 1)$$
$$- 2m\epsilon (k_\nu \tilde{\delta}^\lambda_\mu - k_\mu \tilde{\delta}^\lambda_\nu) \frac{1}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} , \qquad (B.53)$$

and thus, using Eq.(B.17), one obtains

$$C^\lambda_{\mu\nu} = \frac{1}{8} m\epsilon \left[(\tilde{\delta}^\lambda_\nu \bar{k}\cdot\Box - k_\nu \tilde{\partial}^\lambda)\partial_\mu - (\tilde{\delta}^\lambda_\mu \bar{k}\cdot\Box - k_\mu \tilde{\partial}^\lambda)\partial_\nu \right]$$
$$\times \int_{-1}^{1} d\xi \,\{ L_2(Z_\xi) + (1 - Z_\xi)\ln(Z_\xi - 1) \}$$
$$+ \frac{1}{4} m\epsilon (\tilde{\delta}^\lambda_\mu k_\nu - \tilde{\delta}^\lambda_\nu k_\mu) \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} \ln(Z_\xi - 1) . \qquad (B.54)$$

Notice also that

$$(\tilde{\delta}^\lambda_\mu \bar{k}\cdot\Box - k_\mu \tilde{\partial}^\lambda) \int_{-1}^{1} d\xi \,\{ L_2(Z_\xi) + (1 - Z_\xi)\ln(Z_\xi - 1) \} =$$
$$- \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} \left[\tilde{\delta}^\lambda_\mu \bar{k}\cdot(\bar{p} + \xi\bar{k}) - \tilde{p}^\lambda k_\mu \right] \left\{ 1 + (\frac{1}{Z_\xi} - 1)\ln(Z_\xi - 1) \right\} . \qquad (B.55)$$

Therefore, the function u_3 is given by

$$u_3 = \frac{1}{4} m\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi^2} \left\{ (1 - \frac{1}{Z_\xi})\ln(Z_\xi - 1) - 2 \right\} . \qquad (B.56)$$

This expression, together with the scalar functions f_k and f_p allows the definitions, $F^\lambda_k = f_k \tilde{p}^\lambda$ and $F^\lambda_p = f_p \tilde{p}^\lambda$, in Eqs.($B$.51) and ($B$.52) and are sufficient to determined the functions v_1 and v_2. The calculation is straightforward although somewhat tedious. The final form of the functions v_1 and v_2 expressed in terms of the ξ-integral transform can be written as follows: let us define three subsidiary functions of Z_ξ (I_1, I_2, I_3)

$$I_1 = \frac{1}{Z_\xi} \left[1 - (1 - \frac{1}{Z_\xi})\ln(Z_\xi - 1) \right] , \qquad (B.57)$$

$$I_2 = -\frac{2}{Z_\xi^2} + (\frac{1}{Z_\xi^2} - \frac{2}{Z_\xi^3})\ln(Z_\xi - 1) , \qquad (B.58)$$

and

$$I_3 = \frac{1}{Z_\xi} \ln(Z_\xi - 1) \; . \tag{B.59}$$

With these definitions, the functions v_1 and v_2 are given by

$$\begin{aligned} v_1 = & -\frac{1}{8} m\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi} \left[2I_1 + I_3 \right] \\ & + \frac{1}{4} m\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi^2} \left[\bar{p} \cdot \bar{k}(1-\xi) I_2 - \bar{k}^2 \xi (1-\xi) I_2 \right] \end{aligned} \tag{B.60}$$

and

$$\begin{aligned} v_2 = & \; \frac{1}{8} m\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi} \left[2I_1 + I_3 \right] \\ & + \frac{1}{4} m\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi^2} \left[\bar{p} \cdot \bar{k}(1+\xi) I_2 + \bar{k}^2 \xi (1+\xi) I_2 \right] \; . \end{aligned} \tag{B.61}$$

Finally, the equation for $C^\lambda_{\mu\nu\rho}$ is

$$\begin{aligned} \Box^2 C^\lambda_{\mu\nu\rho} &= -2\epsilon \partial^\rho D^\lambda_{\mu\nu\rho} \\ &= -2\epsilon \Bigg\{ \tilde{\partial}^\lambda \left[\frac{p_{1\mu} p_{2\nu} - p_{1\nu} p_{2\mu}}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] \\ & \quad + \partial^\rho \left[\frac{(p_{1\nu} p_{2\rho} - p_{1\rho} p_{2\nu}) \tilde{\delta}^\lambda_\mu + (p_{1\rho} p_{2\mu} - p_{1\mu} p_{2\rho}) \tilde{\delta}^\lambda_\nu}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] \Bigg\} \; . \end{aligned} \tag{B.62}$$

Since only one function needs to be evaluated, one can try to solve the the following 4-vector equation without any loss in generality, namely,

$$\Box^2 (p_1^\mu p_2^\nu \partial^\rho C^\lambda_{\mu\nu\rho}) = -2\epsilon \partial^\rho (p_1^\mu p_2^\nu D^\lambda_{\mu\nu\rho}) \tag{B.63}$$

obtained from Eq.(B.62) by contracting with the 4-vectors p_1^μ and p_2^μ. Equation (B.63) can now be written as

$$\begin{aligned} \nabla\!\!\!/ (p_1^\mu p_2^\nu \partial^\rho C^\lambda_{\mu\nu\rho}) &= -2\epsilon \Bigg\{ \tilde{\partial}^\lambda \left[\frac{(\bar{p}_1^2 \bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2)}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] \\ & \quad + \partial^\rho \left[\tilde{p}^\lambda \, \frac{(\bar{p}_1 \cdot \bar{p}_2) p_{2\rho} - \bar{p}_2^2 p_{1\rho} + (\bar{p}_1 \cdot \bar{p}_2) p_{1\rho} - \bar{p}_1^2 p_{2\rho}}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] \Bigg\} \\ &= -2\epsilon \Bigg\{ \tilde{\partial}^\lambda \left[\frac{4\bar{k}^2 \tilde{p}^2}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] - \partial^\rho \left[\frac{4\bar{k}^2 \tilde{p}^\lambda p_\rho}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} \right] \Bigg\} \\ &= \frac{16\epsilon \bar{k}^2 \tilde{p}^\lambda}{(\bar{p}_1^2 - m^2)(\bar{p}_2^2 - m^2)} = -4\epsilon \bar{k}^2 \tilde{\partial}^\lambda \int_{-1}^{1} \frac{d\xi}{\mu_\xi (Z_\xi - 1)} \; . \end{aligned} \tag{B.64}$$

Using Eq. ($B.12$) one has that

$$p_1^\mu p_2^\nu \partial^\rho C^\lambda_{\mu\nu\rho} = -\frac{1}{2}\epsilon \bar{k}^2 \tilde{p}^\lambda \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi}(1 - \frac{1}{Z_\xi}) \ln(Z_\xi - 1) \; . \tag{B.65}$$

If one compares this equation with Eq.($B.20$), it follows that

$$p_1^\mu p_2^\nu \partial^\rho C^\lambda_{\mu\nu\rho} = -\frac{8}{3}\frac{\bar{k}^2}{m} C^\lambda \; . \tag{B.66}$$

The scalar function v_3 is given by

$$\begin{aligned} \partial^\rho (p_1^\mu p_2^\nu C^\lambda_{\mu\nu\rho}) &= \tilde{\partial}^\lambda \left\{ v_3 \left[\bar{p}_1^2 \bar{p}_2^2 - (\bar{p}_1 \cdot \bar{p}_2)^2 \right] \right\} \\ &\quad + \partial^\rho \left\{ v_3 \left[(\bar{p}_1 \cdot \bar{p}_2)(p_{1\rho} + p_{2\rho}) - (\bar{p}_1^2 p_{2\rho} + \bar{p}_2^2 p_{1\rho}) \right] \right\} \\ &= -8\bar{k}^2 v_3 \tilde{p}^\lambda \end{aligned} \tag{B.67}$$

as terms involving derivatives of v_3 disappear. Thus the function v_3 is related to the u-function by the equation,

$$v_3 = \frac{1}{3}\frac{u}{m} \; , \tag{B.68}$$

or explicitly

$$v_3 = \frac{1}{4}\epsilon \tilde{p}^\lambda \int_{-1}^{1} \frac{d\xi}{\mu_\xi Z_\xi} \left\{ 1 + \frac{\ln(Z_\xi - 1)}{Z_\xi} \right\} \; . \tag{B.69}$$

For reference purposes, all functions are presented in Table B-1. Numerical values for the functions u, u_1, u_2, v_3 and e are presented in Tables B-2 to B-6. The values of the variable x cover the perturbation and inner asymptotic regions.

Table B-1 Perturbation Solutions to the Model Equations

Notation:

$$X_\xi = (\bar{p}+\xi\bar{k})^2, \quad \mu_\xi = (m^2-\bar{k}^2)+\bar{k}^2\xi^2 , \quad Z_\xi = X_\xi/\mu_\xi ,$$

$$I_1 = \frac{1}{Z_\xi}\left[1-(1-\frac{1}{Z_\xi})\ln(Z_\xi-1)\right], \quad I_2 = -\frac{2}{Z_\xi^2}+(\frac{1}{Z_\xi^2}-\frac{2}{Z_\xi^3})\ln(Z_\xi-1) ,$$

$$I_3 = \frac{1}{Z_\xi}\ln(Z_\xi-1) .$$

$$\mathcal{C} = \frac{1}{4}\epsilon\left\{\int_{-1}^{1}\left(1-\frac{1}{Z_\xi}-\frac{2\bar{k}^2}{\mu_\xi Z_\xi}\right)\ln(Z_\xi-1)d\xi \right.$$
$$\left. -\left(1-\frac{m^2}{\bar{p}_1^2}\right)\ln\left(\frac{\bar{p}_1^2}{m^2}-1\right)-\left(1-\frac{m^2}{\bar{p}_2^2}\right)\ln\left(\frac{\bar{p}_2^2}{m^2}-1\right)\right\} \quad (B.70)$$

$$v = \mathcal{C}-\frac{1}{4}\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi}(m^2-\mu_\xi)\left\{1-\left(1-\frac{1}{Z_\xi}\right)\ln(Z_\xi-1)\right\} \quad (B.71)$$

$$e = -\frac{1}{2}\int_{-1}^{1}d\xi\left\{\frac{1}{\mu_\xi(Z_\xi-1)}+\frac{(m^2-\mu_\xi)}{\mu_\xi^2}\left[\frac{1}{Z_\xi(Z_\xi-1)}-\frac{\ln(Z_\xi-1)}{Z_\xi^2}\right]\right\} \quad (B.72)$$

$$u_1 = \frac{1}{4}\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\left\{2+\left[\frac{2}{Z_\xi}-1\right]\ln(Z_\xi-1)\right\}$$
$$+\frac{1}{4}\epsilon\frac{\bar{p}\cdot\bar{k}}{\bar{k}^2}\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\left\{\frac{(Z_\xi-2)}{(Z_\xi-1)}+2\frac{\ln(Z_\xi-1)}{Z_\xi}\right\} \quad (B.73)$$

$$u_2 = \frac{1}{4}\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\left\{2+\left[\frac{2}{Z_\xi}-1\right]\ln(Z_\xi-1)\right\}$$
$$-\frac{1}{4}\epsilon\frac{\bar{p}\cdot\bar{k}}{\bar{k}^2}\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2-\mu_\xi)\left\{\frac{(Z_\xi-2)}{(Z_\xi-1)}+2\frac{\ln(Z_\xi-1)}{Z_\xi}\right\} \quad (B.74)$$

$$v_3 = \frac{1}{4}\epsilon\tilde{p}^\lambda\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi}\left\{1+\frac{\ln(Z_\xi-1)}{Z_\xi}\right\} \quad (B.75)$$

$$u = \frac{3}{4}\epsilon m\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi}\left\{1+\frac{\ln(Z_\xi-1)}{Z_\xi}\right\} . \quad (B.76)$$

Table B-1 (Continued)

$$
\begin{aligned}
v_1 = &-\frac{1}{8}m\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi}\left[2I_1 + I_3\right] \\
&+\frac{1}{4}m\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2}\left[\bar{p}\cdot\bar{k}(1-\xi)I_2 - \bar{k}^2\xi(1-\xi)I_2\right] \ , \qquad (B.77)
\end{aligned}
$$

$$
\begin{aligned}
v_2 = &\ \frac{1}{8}m\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi}\left[2I_1 + I_3\right] \\
&+\frac{1}{4}m\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi^2}\left[\bar{p}\cdot\bar{k}(1+\xi)I_2 + \bar{k}^2\xi(1+\xi)I_2\right] \ . \qquad (B.78)
\end{aligned}
$$

$$
u_3 = \frac{1}{4}m\epsilon\int_{-1}^{1}\frac{d\xi}{\mu_\xi Z_\xi^2}\left\{(1-\frac{1}{Z_\xi})\ln(Z_\xi - 1) - 2\right\} \ . \qquad (B.79)
$$

Table B-2 The Perturbation Functions u, u_1, u_2, v_3 and e as Functions of the Variable y for $x = 1$ and $\bar{k}^2 = 0.1$

y	u ($\times 10^3$)	u_1($\times 10^4$)	u_2($\times 10^4$)	v_3($\times 10^3$)	e ($\times 10^0$)
1.1	-3.0219	-5.4450	5.1248	-1.0073	-0.6688
1.2	-1.9098	-5.4118	5.5385	-0.6366	-0.6167
1.3	-1.3252	-5.1836	5.5005	-0.4417	-0.5797
1.4	-0.9509	-4.9085	5.3192	-0.3170	-0.5500
1.5	-0.6881	-4.6286	5.0860	-0.2294	-0.5249
1.6	-0.4933	-4.3587	4.8372	-0.1644	-0.5031
1.7	-0.3436	-4.1044	4.5889	-0.1145	-0.4838
1.8	-0.2254	-3.8671	4.3486	-0.0751	-0.4665
1.9	-0.1304	-3.6469	4.1197	-0.0435	-0.4508
2.0	-0.0527	-3.4428	3.9037	-0.0176	-0.4365
3.0	0.2891	-2.0523	2.3685	0.0964	-0.3387
4.0	0.3686	-1.3236	1.5404	0.1229	-0.2821
5.0	0.3847	-0.8929	1.0481	0.1282	-0.2441
6.0	0.3802	-0.6161	0.7318	0.1267	-0.2164
7.0	0.3684	-0.4275	0.5166	0.1228	-0.1951
8.0	0.3543	-0.2934	0.3637	0.1181	-0.1781
9.0	0.3398	-0.1948	0.2515	0.1133	-0.1643
10.0	0.3256	-0.1205	0.1670	0.1085	-0.1527
20.0	0.2270	0.1363	-0.1252	0.0757	-0.0927
30.0	0.1755	0.1795	-0.1752	0.0585	-0.0684
40.0	0.1430	0.1907	-0.1885	0.0477	-0.0548
50.0	0.1200	0.1888	-0.1877	0.0400	-0.0460
60.0	0.1027	0.1789	-0.1783	0.0342	-0.0399
70.0	0.0891	0.1648	-0.1644	0.0297	-0.0353
80.0	0.0782	0.1489	-0.1488	0.0261	-0.0317
90.0	0.0693	0.1331	-0.1331	0.0231	-0.0288
100.0	0.0620	0.1182	-0.1183	0.0207	-0.0265
200.0	0.0276	0.0313	-0.0315	0.0092	-0.0150
300.0	0.0165	0.0024	-0.0025	0.0055	-0.0107
400.0	0.0112	-0.0093	0.0091	0.0037	-0.0084
500.0	0.0082	-0.0145	0.0144	0.0027	-0.0069
600.0	0.0064	-0.0171	0.0170	0.0021	-0.0059
700.0	0.0051	-0.0182	0.0182	0.0017	-0.0052
800.0	0.0042	-0.0187	0.0186	0.0014	-0.0046
900.0	0.0035	-0.0188	0.0187	0.0012	-0.0042
1000.0	0.0030	-0.0186	0.0186	0.0010	-0.0038

Table B-3 The Perturbation Functions u, u_1, u_2, v_3 and e as Functions of the Variable y for $x = 10$ and $\overline{k}^2 = 0.1$

y	u ($\times 10^4$)	u_1($\times 10^5$)	u_2($\times 10^5$)	v_3($\times 10^3$)	e ($\times 10^0$)
1.1	4.1158	2.2057	-2.1843	1.3719	-0.1073
1.2	3.9866	2.9492	-2.9287	1.3289	-0.1045
1.3	3.8658	3.4246	-3.4049	1.2886	-0.1020
1.4	3.7525	3.7571	-3.7383	1.2508	-0.0996
1.5	3.6458	3.9982	-3.9802	1.2153	-0.0974
1.6	3.5452	4.1754	-4.1582	1.1817	-0.0953
1.7	3.4500	4.3058	-4.2893	1.1500	-0.0934
1.8	3.3597	4.4006	-4.3848	1.1199	-0.0916
1.9	3.2739	4.4675	-4.4524	1.0913	-0.0899
2.0	3.1922	4.5123	-4.4979	1.0641	-0.0883
3.0	2.5381	4.3514	-4.3420	0.8460	-0.0772
4.0	2.0595	3.7814	-3.7753	0.6865	-0.0730
5.0	1.6313	2.9726	-2.9689	0.5438	-0.0924
6.0	1.2907	2.2549	-2.2533	0.4302	-0.0557
7.0	1.1057	1.9353	-1.9349	0.3686	-0.0447
8.0	0.9760	1.7255	-1.7259	0.3253	-0.0381
9.0	0.8780	1.5713	-1.5722	0.2927	-0.0335
10.0	0.8004	1.4506	-1.4518	0.2668	-0.0301
20.0	0.4456	0.8852	-0.8864	0.1485	-0.0157
30.0	0.3175	0.6699	-0.6708	0.1058	-0.0110
40.0	0.2482	0.5547	-0.5553	0.0827	-0.0085
50.0	0.2036	0.4782	-0.4786	0.0679	-0.0070
60.0	0.1721	0.4193	-0.4196	0.0574	-0.0060
70.0	0.1484	0.3705	-0.3708	0.0495	-0.0052
80.0	0.1301	0.3290	-0.3293	0.0434	-0.0046
90.0	0.1155	0.2935	-0.2937	0.0385	-0.0042
100.0	0.1036	0.2629	-0.2631	0.0345	-0.0038
200.0	0.0489	0.1068	-0.1068	0.0163	-0.0021
300.0	0.0307	0.0535	-0.0536	0.0102	-0.0015
400.0	0.0218	0.0284	-0.0284	0.0073	-0.0011
500.0	0.0165	0.0141	-0.0141	0.0055	-0.0009
600.0	0.0130	0.0051	-0.0051	0.0043	-0.0008
700.0	0.0105	-0.0011	0.0011	0.0035	-0.0007
800.0	0.0086	-0.0055	0.0055	0.0029	-0.0006
900.0	0.0072	-0.0087	0.0087	0.0024	-0.0005
1000.0	0.0061	-0.0112	0.0112	0.0020	-0.0005

Table B-4 The Perturbation Functions u, u_1, u_2, v_3 and e as Functions of the Variable y for $x = 10^3$ and $\overline{k}^2 = 0.1$

y	u ($\times 10^6$)	u_1($\times 10^6$)	u_2($\times 10^6$)	v_3($\times 10^6$)	e ($\times 10^4$)
1.1	3.3953	0.1657	-0.1665	1.1318	-9.6858
1.2	3.2917	0.2229	-0.2235	1.0972	-9.3891
1.3	3.1954	0.2600	-0.2606	1.0651	-9.1132
1.4	3.1055	0.2865	-0.2871	1.0352	-8.8560
1.5	3.0215	0.3062	-0.3068	1.0072	-8.6154
1.6	2.9428	0.3212	-0.3217	0.9809	-8.3899
1.7	2.8687	0.3328	-0.3332	0.9562	-8.1779
1.8	2.7990	0.3417	-0.3421	0.9330	-7.9782
1.9	2.7331	0.3486	-0.3490	0.9110	-7.7897
2.0	2.6709	0.3538	-0.3542	0.8903	-7.6114
3.0	2.1921	0.3625	-0.3628	0.7307	-6.2410
4.0	1.8762	0.3448	-0.3450	0.6254	-5.3369
5.0	1.6496	0.3233	-0.3234	0.5499	-4.6888
6.0	1.4780	0.3027	-0.3028	0.4927	-4.1979
7.0	1.3427	0.2841	-0.2842	0.4476	-3.8114
8.0	1.2331	0.2676	-0.2676	0.4110	-3.4979
9.0	1.1420	0.2529	-0.2529	0.3807	-3.2379
10.0	1.0651	0.2398	-0.2399	0.3550	-3.0182
20.0	0.6579	0.1614	-0.1614	0.2193	-1.8564
30.0	0.4882	0.1248	-0.1248	0.1627	-1.3738
40.0	0.3913	0.1034	-0.1034	0.1304	-1.0994
50.0	0.3269	0.0889	-0.0890	0.1090	-0.9179
60.0	0.2805	0.0781	-0.0782	0.0935	-0.7874
70.0	0.2452	0.0696	-0.0696	0.0817	-0.6886
80.0	0.2176	0.0627	-0.0627	0.0725	-0.6112
90.0	0.1953	0.0569	-0.0569	0.0651	-0.5490
100.0	0.1770	0.0520	-0.0520	0.0590	-0.4980
200.0	0.0904	0.0273	-0.0273	0.0301	-0.2556
300.0	0.0603	0.0183	-0.0183	0.0201	-0.1712
400.0	0.0452	0.0137	-0.0137	0.0151	-0.1286
500.0	0.0361	0.0110	-0.0110	0.0120	-0.1029
600.0	-0.1990	-0.0179	0.0179	-0.0663	-0.1316
700.0	-0.2031	-0.0381	0.0381	-0.0677	-0.1100
800.0	-0.2057	-0.0536	0.0536	-0.0686	-0.0954
900.0	-0.2071	-0.0672	0.0672	-0.0690	-0.0845
1000.0	-0.1942	-21.3921	21.3921	-0.0647	-0.0760

Table B-5 The Perturbation Functions u, u_1, u_2, v_3 and e as Functions of the Variable y for $x = 10^{10}$ and $\overline{k}^2 = 0.1$

y	u ($\times 10^{13}$)	u_1($\times 10^{14}$)	u_2($\times 10^{14}$)	v_3($\times 10^{13}$)	e ($\times 10^{11}$)
1.1	3.3725	1.6408	-1.6408	1.1242	-9.6795
1.2	3.2691	2.2043	-2.2043	1.0897	-9.3825
1.3	3.1729	2.5703	-2.5703	1.0576	-9.1064
1.4	3.0832	2.8312	-2.8312	1.0277	-8.8489
1.5	2.9993	3.0253	-3.0253	0.9998	-8.6082
1.6	2.9206	3.1727	-3.1727	0.9735	-8.3825
1.7	2.8467	3.2859	-3.2859	0.9489	-8.1703
1.8	2.7771	3.3731	-3.3731	0.9257	-7.9705
1.9	2.7114	3.4403	-3.4403	0.9038	-7.7818
2.0	2.6492	3.4914	-3.4914	0.8831	-7.6035
3.0	2.1715	3.5711	-3.5711	0.7238	-6.2323
4.0	1.8563	3.3919	-3.3919	0.6188	-5.3278
5.0	1.6304	3.1761	-3.1761	0.5435	-4.6794
6.0	1.4593	2.9703	-2.9703	0.4864	-4.1884
7.0	1.3246	2.7848	-2.7848	0.4415	-3.8017
8.0	1.2154	2.6200	-2.6200	0.4051	-3.4882
9.0	1.1247	2.4740	-2.4740	0.3749	-3.2281
10.0	1.0482	2.3442	-2.3442	0.3494	-3.0083
20.0	0.6433	1.5661	-1.5661	0.2144	-1.8463
30.0	0.4752	1.2033	-1.2033	0.1584	-1.3639
40.0	0.3798	0.9914	-0.9914	0.1266	-1.0902
50.0	0.3170	0.8491	-0.8491	0.1057	-0.9098
60.0	0.2719	0.7442	-0.7442	0.0906	-0.7804
70.0	0.2379	0.6622	-0.6622	0.0793	-0.6827
80.0	0.2112	0.5961	-0.5961	0.0704	-0.6062
90.0	0.1898	0.5413	-0.5413	0.0633	-0.5448
100.0	0.1722	0.4954	-0.4954	0.0574	-0.4943
200.0	0.0887	0.2634	-0.2634	0.0296	-0.2545
300.0	0.0595	0.1779	-0.1779	0.0198	-0.1707
400.0	0.0447	0.1341	-0.1341	0.0149	-0.1283
500.0	0.0358	0.1075	-0.1075	0.0119	-0.1028
600.0	0.0299	0.0897	-0.0897	0.0100	-0.0857
700.0	0.0256	0.0769	-0.0769	0.0085	-0.0735
800.0	0.0224	0.0674	-0.0674	0.0075	-0.0643
900.0	0.0199	0.0599	-0.0599	0.0066	-0.0572
1000.0	0.0179	0.0539	-0.0539	0.0060	-0.0514

Table B-6 The Perturbation Functions u, u_1, u_2, v_3 and e as Functions of the Variable y for $x = 10^{20}$ and $\bar{k}^2 = 0.1$

y	u ($\times 10^{23}$)	u_1($\times 10^{24}$)	u_2($\times 10^{24}$)	v_3($\times 10^{23}$)	e ($\times 10^{21}$)
1.1	3.3725	1.6408	-1.6408	1.1242	-9.6795
1.2	3.2691	2.2043	-2.2043	1.0897	-9.3825
1.3	3.1729	2.5703	-2.5703	1.0576	-9.1064
1.4	3.0832	2.8312	-2.8312	1.0277	-8.8489
1.5	2.9993	3.0253	-3.0253	0.9998	-8.6082
1.6	2.9206	3.1727	-3.1727	0.9735	-8.3825
1.7	2.8467	3.2859	-3.2859	0.9489	-8.1703
1.8	2.7771	3.3731	-3.3731	0.9257	-7.9705
1.9	2.7114	3.4403	-3.4403	0.9038	-7.7818
2.0	2.6492	3.4914	-3.4914	0.8831	-7.6035
3.0	2.1715	3.5711	-3.5711	0.7238	-6.2323
4.0	1.8563	3.3919	-3.3919	0.6188	-5.3278
5.0	1.6304	3.1761	-3.1761	0.5435	-4.6794
6.0	1.4593	2.9703	-2.9703	0.4864	-4.1884
7.0	1.3246	2.7848	-2.7848	0.4415	-3.8017
8.0	1.2154	2.6200	-2.6200	0.4051	-3.4882
9.0	1.1247	2.4740	-2.4740	0.3749	-3.2281
10.0	1.0482	2.3442	-2.3442	0.3494	-3.0083
20.0	0.6433	1.5657	-1.5657	0.2144	-1.8464
30.0	0.4758	1.1994	-1.1994	0.1586	-1.3654
40.0	0.3818	0.9826	-0.9826	0.1273	-1.0958
50.0	0.3210	0.8378	-0.8378	0.1070	-0.9212
60.0	0.2780	0.7334	-0.7334	0.0927	-0.7980
70.0	0.2460	0.6541	-0.6541	0.0820	-0.7060
80.0	0.2211	0.5917	-0.5917	0.0737	-0.6344
90.0	0.2010	0.5412	-0.5412	0.0670	-0.5770
100.0	0.1846	0.4993	-0.4993	0.0615	-0.5299
200.0	0.1044	0.2899	-0.2899	0.0348	-0.2996
300.0	0.0743	0.2089	-0.2089	0.0248	-0.2132
400.0	0.0582	0.1651	-0.1651	0.0194	-0.1671
500.0	0.0481	0.1372	-0.1372	0.0160	-0.1382
600.0	0.0412	0.1179	-0.1179	0.0137	-0.1182
700.0	0.0361	0.1036	-0.1036	0.0120	-0.1035
800.0	0.0321	0.0926	-0.0926	0.0107	-0.0922
900.0	0.0290	0.0838	-0.0838	0.0097	-0.0833
1000.0	0.0265	0.0767	-0.0767	0.0088	-0.0760

APPENDIX C

ASYMPTOTIC FORMS OF GREEN'S PERTURBATION SOLUTIONS

In order to obtain the asymptotic forms of the perturbation solutions derived in Appendix B, we must observe that the x-dependency of the ξ-integral transform occurs in the function $X_\xi = \bar{p}^2 + 2\bar{p}\cdot\bar{k} + \bar{k}^2$. In the asymptotic region,

$$\bar{p}^2 \sim xy; \qquad \bar{p}\cdot\bar{k} = \tfrac{1}{2}x\sqrt{y^2-1}\ . \tag{C.1}$$

Consequently, X_ξ can be approximated by the expression

$$X_\xi \sim x\Theta(\xi)\ , \tag{C.2}$$

where

$$\Theta(\xi) = y + \xi\sqrt{y^2-1}\ . \tag{C.3}$$

Notice also that $Z_\xi - 1$ can be written in the form

$$Z_\xi - 1 = \left(\frac{x}{\bar{k}^2}\right)\frac{\Theta(\xi)}{\xi^2+\chi^2}\ , \tag{C.4}$$

where

$$\chi = \sqrt{\frac{m^2}{\bar{k}^2} - 1}\ , \tag{C.5}$$

and thus

$$\ln(Z_\xi - 1) \sim \ln\frac{x}{\bar{k}^2} + \ln\Theta(\xi) - \ln(\xi^2+\chi^2)\ . \tag{C.6}$$

To obtain a good approximation of the ξ-integrals, in the asymptotic region, it is necessary to keep only those terms in the integrand that have the smallest power in x^{-1}. We must be careful, not to neglect terms such as Z_ξ^{-1}

against $Z_\xi^{-1} \ln Z_\xi$ due to the slow variation of the logarithm. To avoid confusion between the *perturbation solutions* in the asymptotic region and the actual *asymptotic solutions*, we will use the superscript "pert" when we refer to the former solutions.

Using Eq.(B.20), (See also Integral Table at the end of the chapter) it follows that

$$u^{(pert)} \sim \frac{3}{4} m\epsilon \int_{-1}^{1} \frac{d\xi}{X_\xi} = \frac{3}{2} m\epsilon \frac{\ln z}{x\sqrt{y^2-1}} \ . \tag{C.7}$$

This expression is not accurate enough in the asymptotic region due to the fact that in obtaining the perturbation solutions we set $A_1 \sim A_2 \sim m$. However, by Eq.(3.61), $A_1 \sim A_2 \sim x^{-\eta}$. It has been shown[23,24] that an extended solution can be constructed if we replace the mass m by the average of A_1 and A_2. Assuming this to be valid, then

$$u^{(pert)} = \frac{3}{4}(A_1 + A_2)\epsilon \frac{\ln z}{x\sqrt{y^2-1}} \ . \tag{C.8}$$

The form of the function $v^{(pert)}$ in the asymptotic region can be obtained from Eqs.(B.30) and (B.39). From Eq.(B.30), we see that it is possible to approximate the function $\mathcal{C}$ by the form

$$\mathcal{C} \sim \frac{1}{4}\epsilon \int_{-1}^{1} \ln(Z_\xi - 1) d\xi - \frac{1}{4}\epsilon \left[\ln(\frac{\bar{p}_1^2}{m^2}) + \ln(\frac{\bar{p}_2^2}{m^2})\right] \ , \tag{C.9}$$

and therefore,

$$v^{(pert)} \sim 1 + \mathcal{C} = 1 + \frac{1}{2}\epsilon \left\{ 1 - 2\chi \tan^{-1}(1/\chi) + \frac{y \ln z}{\sqrt{y^2-1}} \right\} \ . \tag{C.10}$$

Notice that the largest term in $v^{(pert)}$ is x-independent; all other terms being of order $\mathcal{O}(1/x)$. From Eq.(B.33) it also follows that the function $e^{(pert)}$ is given by

$$e^{(pert)} \sim -\frac{1}{2} \int_{-1}^{1} \frac{d\xi}{X_\xi} \sim -\frac{\ln z}{x\sqrt{y^2-1}} \ . \tag{C.11}$$

The form of the functions $u_1^{(pert)}$ and $u_2^{(pert)}$ can be extracted from Eqs.(B.46) and (B.47). It is clear that the largest contribution to these functions comes from the integrals being multiplied by $\bar{p} \cdot \bar{k}$. From these equations, it follows that there is a simple relation between the two functions, *i.e.*

$$u_1^{(pert)} = -u_2^{(pert)} \sim \frac{1}{4}\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi^2 Z_\xi^2}(m^2 - \mu_\xi) \, . \qquad (C.12)$$

This integral is elementary (see Table C-1), and hence

$$u_1^{(pert)} = -u_2^{(pert)} \sim \frac{1}{2}\epsilon \left[\frac{y \ln z}{(y^2-1)} - \frac{1}{\sqrt{y^2-1}} \right] \, . \qquad (C.13)$$

We now turn to the problem of finding the form of the functions, $v_1^{(pert)}, v_2^{(pert)}$ and $u_3^{(pert)}$, in the asymptotic region. These functions are particularly difficult to calculate since we must split the terms containing the function $\ln(Z_\xi - 1)$ as prescribed in Eq.(C.6). Also a test of these functions by substitution in their corresponding differential equations, show that there *is not* a simple assumption we can make regarding the transition of the mass m from its perturbation value to its asymptotic form.

The function $u_3^{(pert)}$ given in Eq.(B.56) has the following form in the asymptotic region,

$$u_3^{(pert)} \sim -\frac{1}{4}\epsilon \int_{-1}^{1} \frac{d\xi}{\mu_\xi^2 Z_\xi^2} \left[2 + \ln(Z_\xi - 1)\right] \, . \qquad (C.14)$$

Using Eq.(C.6) and the integrals found in Table C-1 it is possible to write $u_3^{(pert)}$ in a somewhat complex form. To this end, let us define the functions τ_1 and τ_2 by the equations

$$\tau_1 = \frac{y \ln z}{\sqrt{y^2-1}} \quad \text{and} \quad \tau_2 = \frac{\bar{k}^2}{m^2(y^2-1) + \bar{k}^2} \, . \qquad (C.15)$$

Using these definitions, Eq.(C.14) integrates into the form,

$$u_3^{(pert)} \sim \frac{1}{2}\epsilon \left[1 - \ln(x/m^2) + \tau_1 + 2\tau_2[\chi \tan^{-1}(1/\chi) - \tau_1]\right]/x^2 \, , \qquad (C.16)$$

where χ is defined in Eq.(C.5).

The form of $v_1^{(pert)}$ in the asymptotic region can be found in an analogous manner from Eq.(B.60). In this region we have

$$v_1^{(pert)} \sim \frac{1}{8} m\epsilon \int_{-1}^{1} \Big\{ - \frac{1}{\mu_\xi Z_\xi}[2 - \ln(Z_\xi - 1)] \\ - \frac{2\bar{p}\cdot\bar{k}}{\mu_\xi^2 Z_\xi^2}(1-\xi)[2 - \ln(Z_\xi - 1)] \Big\} . \qquad (C.17)$$

Using Eq.(C.6) and defining the integrals,

$$I_1 = \int_{-1}^{1} d\xi \Big\{ -\frac{2}{X_\xi} + \frac{1}{X_\xi}\left[\ln(x/\bar{k}^2) + \ln\Theta(\xi)\right] - \frac{4\bar{p}\cdot\bar{k}}{X_\xi^2}(1-\xi) \\ + \frac{2\bar{p}\cdot\bar{k}}{X_\xi^2}(1-\xi)\left[\ln(x/\bar{k}^2) + \ln\Theta(\xi)\right] \Big\} \qquad (C.18)$$

and

$$I_2 = \int_{-1}^{1} d\xi \ln(\xi^2 + \chi^2) \left\{ \frac{1}{X_\xi} + \frac{2\bar{p}\cdot\bar{k}}{X_\xi^2}(1-\xi) \right\} , \qquad (C.19)$$

it is possible to write Eq.(C.17) in the form,

$$v_1^{(pert)} \sim \frac{1}{8} m\epsilon (I_1 - I_2) . \qquad (C.20)$$

Using Table C-1, it is found that the integral I_1 can be written in the form

$$I_1 = \left[I_{11} - 2\bar{p}\cdot\bar{k}(I_{13} - I_{14})\right]\left[\ln(x/m^2) - 2\right] + 2\bar{p}\cdot\bar{k} I_{15} , \qquad (C.21)$$

where

$$I_{11} = \int_{-1}^{1} \frac{d\xi}{X_\xi} = \frac{2\ln z}{x\sqrt{y^2-1}}, \quad I_{12} = \int_{-1}^{1} \frac{d\xi}{X_\xi}\ln\Theta(\xi) = \frac{1}{x}\int_{-1}^{1} d\xi \frac{\ln\Theta(\xi)}{\Theta(\xi)} = 0 , \qquad (C.22)$$

$$I_{13} = \int_{-1}^{1} \frac{\xi d\xi}{X_\xi^2} = \frac{2}{x^2}\left[\frac{\ln z}{(y^2-1)} - \frac{y}{\sqrt{y^2-1}}\right] , \quad I_{14} = \int_{-1}^{1} \frac{d\xi}{X_\xi^2} = \frac{2}{x^2} , \qquad (C.23)$$

and

$$I_{15} = \int_{-1}^{1} \frac{(1-\xi)\ln\Theta(\xi)d\xi}{X_\xi^2} = \frac{2}{x^2\sqrt{y^2-1}}\left[z - (z + \sqrt{y^2-1})\ln z\right] . \qquad (C.24)$$

Substituting Eqs.(C.22) to (C.24) into Eq.(C.21) gives

$$I_1 = -\frac{2z}{x}\left\{1 + \tau_1 - \frac{\ln(x/\bar{k}^2)}{\sqrt{y^2-1}}\right\}. \qquad (C.25)$$

The integral I_2 given in Eq.(C.19) can be split into three integrals of the form,

$$I_2 = \frac{1}{x}\left[I_{21} + I_{22}\sqrt{y^2-1} - I_{23}\sqrt{y^2-1}\right] \ , \qquad (C.26)$$

where

$$I_{21} = \int_{-1}^{1} \frac{\ln(\xi^2+\chi^2)d\xi}{\Theta(\xi)} \ , \qquad (C.27)$$

$$I_{22} = \int_{-1}^{1} \frac{\ln(\xi^2+\chi^2)d\xi}{\Theta^2(\xi)} \ , \qquad (C.28)$$

and

$$I_{23} = \int_{-1}^{1} \frac{\xi\ln(\xi^2+\chi^2)d\xi}{\Theta^2(\xi)} \ . \qquad (C.29)$$

The integral in Eq.(C.27) cannot be expressed in terms of a finite number of elementary functions. Fortunately, it will not be necessary to evaluate this integral in order to define $v_1^{(pert)}$ (or $v_2^{(pert)}$) since the integral I_{23} can be expressed in terms of the integral I_{21}. To show this, we must integrate by parts Eq.(C.29), *i.e.*

$$I_{23} = \xi I_{22}\Big|_{\xi=-1}^{\xi=+1} - \int_{-1}^{1} I_{22}d\xi \ . \qquad (C.30)$$

In general, we have

$$G(t) = \int \frac{\ln(t^2+a^2)dt}{(ct+d)^2} = -\frac{1}{c}\frac{\ln(t^2+a^2)}{(ct+d)^2} + \frac{2}{c}\int \frac{tdt}{(t^2+a^2)(ct+d)} \ , \qquad (C.31)$$

and

$$\int \frac{tdt}{(t^2+a^2)(ct+d)} = \frac{1}{a^2c^2+d^2}\left\{\frac{1}{2}a\ln(t^2+a^2) - d\ln(ct+d) + ac\tan^{-1}(t/a)\right\} \ . \qquad (C.32)$$

Setting $a = \chi$, $c = \sqrt{y^2-1}$ and $d = y$ in Eq.(C.31) and evaluating the integral gives

$$I_{22} = 2\ln(1+\chi^2) + 4\tau_2[\chi\tan^{-1}(1/\chi) - \tau_1] \tag{C.33}$$

where τ_1 and τ_2 are defined in Eq.(C.14). To complete the evaluation of the integral I_{22}, we must integrate once again Eq.(C.31). This integration is elementary, and yields the result,

$$\begin{aligned}\int_{-1}^{1} d\xi \int_0^{\xi} dt \frac{\ln(t^2+a^2)dt}{(ct+d)^2} &= -\frac{I_{21}}{\sqrt{y^2-1}} \\ &+ \frac{2y\tau_2}{\sqrt{y^2-1}}\left\{\ln(1+\chi^2) + 2\chi\tan^{-1}(1/\chi) - 2\tau_2\right\} .\end{aligned} \tag{C.34}$$

Multiplying Eq.(C.31) by t and evaluating the expression at $t = \pm 1$ gives

$$tG(t)\Big|_{t=-1}^{t=+1} = -2\frac{m^2}{\bar{k}^2}\tau_2 y\sqrt{y^2-1}\,\ln(m^2/\bar{k}^2) . \tag{C.35}$$

Using these results, we find that the integral I_2 can be written in the form,

$$I_2 = \frac{\sqrt{y^2-1}}{x}[I_{22} - \tau_3(y)] , \tag{C.36}$$

where the function τ_3 is defined by

$$\tau_3(y) = -\frac{2y\tau_2}{\sqrt{y^2-1}}\left\{\ln(m^2/\bar{k}^2)/\tau_2 + 2\chi\tan^{-1}(1/\chi) - 2\tau_1\right\} . \tag{C.37}$$

Notice that the integral I_{21} has disappeared.

Finally, substituting Eqs.(C.36) and (C.25) into Eq.(C.20) yields the desired result for the function $v_1^{(pert)}$, namely

$$v_1^{(pert)} = -\frac{1}{4}\epsilon H(x,y)\frac{z}{x} , \tag{C.38}$$

where

$$H(x,y) = 1 - \ln(x/m^2) + \frac{y\ln z}{\sqrt{y^2-1}} + 2\tau_2[\chi\tan^{-1}(1/\chi) - \tau_1] . \tag{C.39}$$

A similar procedure can be used to show that the function $v_2^{(pert)}$ is given by the expression,

$$v_2^{(pert)} = +\frac{1}{4}\epsilon H(x,y)\frac{z^{-1}}{x} . \tag{C.40}$$

For convenience, all asymptotic forms are presented in Table C-2.

Table C-1 Table of Useful Integrals.

Notations:

$$a = xy; \quad b = x\sqrt{y^2-1}; \quad \alpha = (\frac{xy}{m^2} - 1); \quad \beta = \frac{x\sqrt{y^2-1}}{m^2} \; ;$$

$$S = \frac{(xz - m^2)}{(xz^{-1} - m^2)}; \quad R = (xz - m^2)(xz^{-1} - m^2)/m^4; \quad \chi = \sqrt{\frac{m^2}{\bar{k}^2} - 1} \; ;$$

$$\tau_2 = \frac{\bar{k}^2}{m^2(y^2-1) + \bar{k}^2} \; .$$

1. $$\int_{-1}^{1} \frac{d\xi}{(a+b\xi)} = \frac{2\ln z}{x\sqrt{y^2-1}} \; .$$
2. $$\int_{-1}^{1} \frac{d\xi}{(a+b\xi)} = \frac{2}{x^2} \; .$$
3. $$\int_{-1}^{1} \frac{\xi d\xi}{(a+b\xi)} = \frac{2}{x\sqrt{y^2-1}} \left[1 - \frac{y\ln z}{\sqrt{y^2-1}}\right] \; .$$
4. $$\int_{-1}^{1} \frac{\xi^2 d\xi}{(a+b\xi)} = \frac{-2y}{x(y^2-1)} \left[1 - \frac{y\ln z}{\sqrt{y^2-1}}\right] \; .$$
5. $$\int_{-1}^{1} \frac{d\xi}{(a+b\xi)^2} = \frac{2}{x^2} \; .$$
6. $$\int_{-1}^{1} \frac{\xi d\xi}{(a+b\xi)^2} = \frac{2}{x^2} \left[\frac{\ln z}{(y^2-1)} - \frac{y}{\sqrt{y^2-1}}\right] \; .$$
7. $$\int_{-1}^{1} \frac{\xi^2 d\xi}{(a+b\xi)} = \frac{2}{x^2(y^2-1)} \left[1 + y^2 - y\ln z\right] \; .$$
8. $$\int_{-1}^{1} \frac{d\xi}{(a+b\xi)^3} = \frac{2y}{x^3} \; .$$
9. $$\int_{-1}^{1} \frac{d\xi}{(a+b\xi)(\alpha+\beta\xi)} = \frac{-2}{x^2} + \frac{1}{m^2 x\sqrt{y^2-1}} \left[\ln S - 2\ln z\right] \; .$$
10. $$\int_{-1}^{1} \ln(\alpha+\beta\xi) d\xi = -2 + \frac{2y\ln z}{x\sqrt{y^2-1}} + 2\ln(x/m^2) \; .$$

Table C-1 (Continued)

10. $\displaystyle\int_{-1}^{1} \frac{\ln(\alpha+\beta\xi)d\xi}{(a+b\xi)} \sim \frac{1}{2}\frac{[\ln(xz/m^2)]^2 - [\ln(xz^{-1}/m^2)]^2}{x\sqrt{y^2-1}} \ .$

11. $\displaystyle\int_{-1}^{1} \frac{\ln(a+b\xi)d\xi}{(a+b\xi)} = 0 \ .$

12 – $\displaystyle\int_{-1}^{1} \frac{\ln(a+b\xi)d\xi}{(a+b\xi)^2} = 2\left[1 - \frac{y\ln z}{\sqrt{y^2-1}}\right] \ .$

13. $\displaystyle\int_{-1}^{1} \frac{\xi\ln(a+b\xi)d\xi}{(a+b\xi)^2} = -\frac{2y}{\sqrt{y^2-1}} + 2\ln z\left[1 + \frac{1}{\sqrt{y^2-1}}\right] \ .$

14. $\displaystyle\int_{-1}^{1} \frac{\ln(\xi^2+\chi^2)d\xi}{(a+b\xi)^2} = 2\ln(1+\chi^2) + 4\tau_2\left[\chi\tan^{-1}(1/\chi) - \frac{y\ln z}{\sqrt{y^2-1}}\right] \ .$

Table C-2 Asymptotic Forms of the Perturbation Solutions

Notation:

$$\tau_1 = \frac{y \ln z}{\sqrt{y^2-1}}; \quad \tau_2 = \frac{\bar{k}^2}{m^2(y^2-1)+\bar{k}^2} ;$$

$$\tau_3(y) = -\frac{2y\tau_2}{\sqrt{y^2-1}} \left\{ \ln(m^2/\bar{k}^2)/\tau_2 + 2\chi \tan^{-1}(1/\chi) - 2\tau_1 \right\} ;$$

$$H(x,y) = 1 - \ln(x/m^2) + \frac{y \ln z}{\sqrt{y^2-1}} - 2\tau_2[\tau_1 - \chi \tan^{-1}(1/\chi)] ; \quad \chi = \sqrt{\frac{m^2}{\bar{k}^2} - 1} .$$

$$v^{(pert)} \sim 1 + \mathcal{C} = 1 + \frac{1}{4}\epsilon \left\{ 2 - 4\chi \tan^{-1}(1/\chi) + \frac{2y \ln z}{\sqrt{y^2-1}} \right\} . \tag{C.41}$$

$$e^{(pert)} \sim -\frac{\ln z}{x\sqrt{y^2-1}} \tag{C.42}$$

$$u_1^{(pert)} \sim +\frac{1}{2}\epsilon \left[\frac{y \ln z}{(y^2-1)} - \frac{1}{\sqrt{y^2-1}} \right] . \tag{C.43}$$

$$u_2^{(pert)} \sim -\frac{1}{2}\epsilon \left[\frac{y \ln z}{(y^2-1)} - \frac{1}{\sqrt{y^2-1}} \right] . \tag{C.44}$$

$$v_3^{(pert)} \sim \frac{1}{2}\epsilon \frac{\ln z}{x\sqrt{y^2-1}} . \tag{C.45}$$

$$u^{(pert)} \sim \frac{3}{4}(A_1 + A_2)\epsilon \frac{\ln z}{x\sqrt{y^2-1}} . \tag{C.46}$$

$$v_1^{(pert)} = -\frac{1}{4}\epsilon H(x,y) \frac{z^{-1}}{x} , \tag{C.47}$$

$$v_2^{(pert)} = +\frac{1}{4}\epsilon H(x,y) \frac{z}{x} . \tag{C.48}$$

$$u_3^{(pert)} \sim \frac{1}{2}\epsilon H(x,y)/x^2 , \tag{C.49}$$

APPENDIX D
ASYMPTOTIC BOUNDARY CONDITIONS

To obtain the boundary conditions to be used with the differential equations derived from the model equation, we use a technique developed and used by A. Broyles[20] in connection with the determination of the electron propagator functions. This technique is equivalent to the use of Green's theorem by which it is possible to convert certain volume integrals into surface integrals.

In order to find the boundary conditions for the vertex function, we start with the definition of the vertex integral equation as given in Eq.(2.34), namely,

$$\underline{\Gamma}^{\lambda}(\bar{p}_1,\bar{p}_2)=\underline{\gamma}_0^{\lambda}+\frac{i\epsilon}{Z_3(2\pi)^2}\int D_{\mu\nu}(\bar{p}-\bar{q})\underline{\gamma}^{\mu}\underline{F}^{\lambda\nu}(\bar{q}_1,\bar{q}_2)d^4q \qquad (D.1)$$

where

$$\epsilon=\frac{Z_3e_0^2}{(2\pi)^2}=\frac{\alpha}{\pi},\quad \partial_\nu\equiv\partial/\partial p^\nu\ , \qquad (D.2)$$

$$\underline{F}^{\lambda\nu}(\bar{q}_1,\bar{q}_2)\sim-\underline{S}(\bar{q}_1)\underline{\Gamma}^{\lambda}(\bar{q}_1,\bar{q}_2)\underline{S}(\bar{q}_2)\underline{\gamma}^{\nu}\ , \qquad (D.3)$$

and

$$D_{\mu\nu}(\bar{k})=Z_3\left[-g_{\mu\nu}+k_\mu k_\nu\bar{k}^{-2}\right]\bar{k}^{-2}\ . \qquad (D.4)$$

We have also defined the 4-vectors $\bar{p}_1=\bar{p}+\bar{k}$, $\bar{p}_2=\bar{p}-\bar{k}$ which represent the outgoing and incoming electron momenta respectively. Substituting Eq.(D.4) into Eq.(D.1) gives

$$\underline{\Gamma}^{\lambda}(\bar{p}_1,\bar{p}_2)=\underline{\gamma}^{\lambda}+\frac{i\epsilon}{(2\pi)^2}\int\{-\frac{\underline{\gamma}_\nu\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2}+\frac{(\not{p}-\not{q})(p_\nu-q_\nu)\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^4}\}d^4q\ . \qquad (D.5)$$

Using the identity,

$$p_\mu p_\nu p^{-4} = -\frac{1}{4}\partial_\mu\partial_\nu \ln p^2 + \frac{1}{2}g_{\mu\nu}p^{-2} \ , \qquad (D.6)$$

the last term in Eq.(D.5) can be written as

$$\int \frac{(\not{p}-\not{q})(p_\nu - q_\nu)\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^4} d^4q = \frac{1}{2}\underline{\gamma}^\mu g_{\mu\nu} \int \frac{\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2} - \frac{1}{4}\Im \qquad (D.7)$$

where

$$\Im = \not{\nabla}\partial_\nu \int \ln(\bar{p}-\bar{q})^2 \underline{F}^{\lambda\nu}(\bar{q}_1, \bar{q}_2) d^4q \ . \qquad (D.8)$$

Substituting Eq.(D.7) into (D.5) yields

$$\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2) = \underline{\gamma}^\lambda - \frac{1}{2}\frac{i\epsilon}{(2\pi)^2} \int \frac{\underline{\gamma}_\nu \underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2} d^4q - \frac{1}{4}\frac{i\epsilon}{(2\pi)^2}\Im \ . \qquad (D.9)$$

We can apply $\not{\nabla}$ to Eq.(D.9) with the result,

$$\not{\nabla}\underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2) = -\frac{1}{2}\frac{i\epsilon}{(2\pi)^2}\not{\nabla}\underline{\gamma}_\nu \int \frac{\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2} - \frac{1}{4}\frac{i\epsilon}{(2\pi)^2}\not{\nabla}\Im \ . \qquad (D.10)$$

We would like to expresss the integral $\Im$ so that the $\bar{p}$-dependency appears only in the form $(\bar{p}-\bar{q})^{-2}$. Notice that from Eq.(D.8),

$$\begin{aligned} \not{\nabla}\Im &= -\int \left[\Box_q^2 \partial_{q_\nu} \ln(\bar{p}-\bar{q})^2\right] \underline{F}^{\lambda\nu} d^4q \\ &= -\int \partial_{q_\nu} \left\{ [\Box_q^2 \ln(\bar{p}-\bar{q})^2] \underline{F}^{\lambda\nu} \right\} d^4q \\ &\quad + \int \left\{ \Box_q^2 \ln(\bar{p}-\bar{q})^2 \right\} \partial_{q_\nu} \underline{F}^{\lambda\nu} d^4q \ , \end{aligned} \qquad (D.11)$$

where $\Box_q^2$ stands for the D'Alambertian operator in the momentum q (see also Eq.(2.35)) and $\partial_q{}^\nu = \partial/\partial q_\nu$. In general, a subscript q will be used to denote derivatives with respect to q. The first integral in Eq.(D.11) can easily be obtained and is given by

$$\int \partial_{q_\nu} \left\{ [\Box_q^2 \ln(\bar{p}-\bar{q})^2] \underline{F}^{\lambda\nu} \right\} d^4q = \oint \left\{ [\Box_q^2 \ln(\bar{p}-\bar{q})^2] \underline{F}^{\lambda\nu} \right\} d\sigma_\nu \qquad (D.12)$$

where $d\sigma_\nu$ is the outward drawn normal to the volume d^4q. The identity,

$$\Box^2 \ln \bar{p}^2 = 4\bar{p}^{-2} \ , \tag{D.13}$$

can be used to show that

$$\not{\nabla}\Im = -4\oint \frac{\underline{F}^{\lambda\nu} d\sigma_\nu}{(\bar{p}-\bar{q})^2} + 4\int \frac{\partial_{q\,\nu}\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2} d^4q \ . \tag{D.14}$$

Also, from the definition of $\not{\nabla}$, it follows that

$$\begin{aligned}\not{\nabla}\underline{\gamma}_\nu \int \frac{\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2} &= \ \underline{\gamma}^\alpha\underline{\gamma}_\nu \int [\partial_\alpha(\bar{p}-\bar{q})^{-2}]\underline{F}^{\lambda\nu} d^4q \\ &= -\underline{\gamma}^\alpha\underline{\gamma}_\nu \int [\partial_{q\,\alpha}(\bar{p}-\bar{q})^{-2}]\underline{F}^{\lambda\nu} d^4q \ .\end{aligned} \tag{D.15}$$

It is possible to transform Eq.$(D.15)$ using the identity,

$$\partial_{q\,\alpha}[(\bar{p}-\bar{q})^{-2}\underline{F}^{\lambda\nu}] = [\partial_{q\,\alpha}(\bar{p}-\bar{q})^{-2}]\underline{F}^{\lambda\nu} + (\bar{p}-\bar{q})^{-2}\partial_{q\,\alpha}\underline{F}^{\lambda\nu} \ . \tag{D.16}$$

Using Eq.$(D.16)$ in Eq.$(D.15)$ yields the result

$$\not{\nabla}\underline{\gamma}_\nu \int \frac{\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2} = -\underline{\gamma}^\alpha\underline{\gamma}_\nu \oint \frac{\underline{F}^{\lambda\nu} d\sigma_\alpha}{(\bar{p}-\bar{q})^2} + \underline{\gamma}^\alpha\underline{\gamma}_\nu \int \frac{\partial_{q\,\alpha}\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2} \ . \tag{D.17}$$

Substituting Eq.$(D.17)$ and $(D.14)$ into Eq.$(D.10)$ we obtain

$$\begin{aligned}\not{\nabla}\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) &= \frac{1}{2}\frac{i\epsilon}{(2\pi)^2}\left\{\oint \frac{d\not{\sigma}\underline{\gamma}_\nu\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2} - \int \frac{\not{\nabla}\underline{\gamma}_\nu\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2}\right\} \\ &+ \frac{i\epsilon}{(2\pi)^2}\left\{\oint \frac{\underline{F}^{\lambda\nu} d\sigma_\nu}{(\bar{p}-\bar{q})^2} - \int \frac{\partial_{q\,\nu}\underline{F}^{\lambda\nu} d^4q}{(\bar{p}-\bar{q})^2}\right\} ,\end{aligned} \tag{D.18}$$

or, grouping all integrals over the volume d^4q,

$$\begin{aligned}\not{\nabla}\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) &= -\frac{i\epsilon}{(2\pi)^2}\int (\bar{p}-\bar{q})^{-2}\left[\tfrac{1}{2}\not{\nabla}_q\underline{\gamma}_\nu\underline{F}^{\lambda\nu} + \partial_{q\,\nu}\underline{F}^{\lambda\nu}\right] d^4q \\ &+ \frac{i\epsilon}{(2\pi)^2}\left\{\frac{1}{2}\oint \frac{d\not{\sigma}\underline{\gamma}_\nu\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2} + \oint \frac{\underline{F}^{\lambda\nu} d\sigma_\nu}{(\bar{p}-\bar{q})^2}\right\} .\end{aligned} \tag{D.19}$$

We can reduce Eq.$(D.18)$ to a differential equation by using the identity

$$\Box^2 p^{-2} = i(2\pi)^2 \delta^4(\bar{p}) \ . \tag{D.20}$$

If we exclude an infinitesimal volume around the point $\bar{p} = \bar{q}$, there will be no contributions from the surface integrals in Eq.(D.19). Hence, if we apply the D'Alambertian $\Box^2$ to Eq.(D.19), we obtain

$$\Box^2 \not{\nabla} \underline{\Gamma}^\lambda(\bar{p}_1, \bar{p}_2) = \epsilon(\partial_\nu \underline{F}^{\lambda\nu} + \tfrac{1}{2} \not{\nabla} \underline{\gamma}_\mu \underline{F}^{\lambda\nu}) \ . \tag{D.21}$$

Notice that this is the same equation obtained in Chapter II by an independent method (see Eq.(2.50)).

In order to obtain the boundary conditions to the differential equation given in Eq.(D.21), we integrate Eq.(D.8) by parts excluding the point ($\bar{p} = \bar{q}$) to obtain

$$\begin{aligned} \Im &= -\not{\nabla} \int [\partial_{q_\nu} \ln(\bar{p} - \bar{q})^2] \underline{F}^{\lambda\nu} d^4 q \\ &= -\not{\nabla} \oint \ln(\bar{p} - \bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu + \not{\nabla} \int \ln(\bar{p} - \bar{q})^2 \partial_{q_\nu} \underline{F}^{\lambda\nu} d^4 q \ . \end{aligned} \tag{D.22}$$

We can rearrange Eq.(D.21) to obtain the form,

$$\epsilon \partial_\nu \underline{F}^{\lambda\nu} = \Box^2 \not{\nabla} \underline{\Gamma}^\lambda - \tfrac{1}{2} \epsilon \not{\nabla} \underline{\gamma}_\nu \underline{F}^{\lambda\nu} \ . \tag{D.23}$$

Substituting Eq.(D.23) into Eq.(D.22) gives

$$\begin{aligned} \epsilon \Im = &- \epsilon \not{\nabla} \oint \ln(\bar{p} - \bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu + \not{\nabla} \int \ln(\bar{p} - \bar{q})^2 \Box_q^2 \not{\nabla}_q \underline{\Gamma}^\lambda d^4 q \\ &+ \tfrac{1}{2} \epsilon \not{\nabla} \int \ln(\bar{p} - \bar{q})^2 \not{\nabla}_q \underline{\gamma}_\nu \underline{F}^{\lambda\nu} d^4 q \ . \end{aligned} \tag{D.24}$$

Integrating by parts the last two terms in Eq.(D.24) gives

$$\begin{aligned} \epsilon \Im = &- \epsilon \not{\nabla} \oint \ln(\bar{p} - \bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu + \not{\nabla} \oint \ln(\bar{p} - \bar{q})^2 \Box_q^2 d\not{\sigma} \underline{\Gamma}^\lambda \\ &- \not{\nabla} \int [\not{\nabla}_q \ln(\bar{p} - \bar{q})^2] \Box_q^2 \underline{\Gamma}^\lambda d^4 q - \tfrac{1}{2} \epsilon \not{\nabla} \oint \ln(\bar{p} - \bar{q})^2 d\not{\sigma} \underline{\gamma}_\nu \underline{F}^{\lambda\nu} \\ &+ \tfrac{1}{2} \epsilon \not{\nabla} \int [\not{\nabla}_q \ln(\bar{p} - \bar{q})^2] \underline{\gamma}_\nu \underline{F}^{\lambda\nu} d^4 q \ . \end{aligned} \tag{D.25}$$

Replacing $\not\nabla_q$ with $-\not\nabla$ and rearranging terms gives

$$
\begin{aligned}
\epsilon\Im = & \int [\Box_q^2 \ln(\bar{p}-\bar{q})^2] \Box_q^2 \underline{\Gamma}^\lambda d^4q - \tfrac{1}{2}\epsilon \int [\Box_q^2 \ln(\bar{p}-\bar{q})^2] \underline{\gamma}_\nu \underline{F}^{\lambda\nu} d^4q \\
& - \epsilon \not\nabla \oint \ln(\bar{p}-\bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu + \not\nabla \oint \ln(\bar{p}-\bar{q})^2 d\not\sigma \Box_q^2 \underline{\Gamma}^\lambda \\
& - \tfrac{1}{2}\epsilon \not\nabla \oint \ln(\bar{p}-\bar{q})^2 d\not\sigma \underline{\gamma}_\nu \underline{F}^{\lambda\nu} \ . \qquad (D.26)
\end{aligned}
$$

We can now make use of Eq.(D.13) and substitute the previous expression into Eq.(D.9) to obtain

$$
\begin{aligned}
\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) = \underline{\gamma}^\lambda - \frac{1}{4}\frac{i}{(2\pi)^2} \Big\{ & \ 4 \int \frac{\Box_q^2 \underline{\Gamma}^\lambda d^4q}{(\bar{p}-\bar{q})^2} - \epsilon \not\nabla \oint \ln(\bar{p}-\bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu \\
& + \not\nabla \oint \ln(\bar{p}-\bar{q})^2 d\not\sigma \Box_q^2 \underline{\Gamma}^\lambda \\
& - \tfrac{1}{2} \not\nabla \oint \ln(\bar{p}-\bar{q})^2 d\not\sigma \underline{\gamma}_\nu \underline{F}^{\lambda\nu} \Big\} \ . \qquad (D.27)
\end{aligned}
$$

At this stage, it is possible to partially integrate the first volume integral excluding the point $\bar{p}=\bar{q}$ and therefore

$$
\begin{aligned}
\int_{\bar{p}\neq\bar{q}} \frac{\Box_q^2 \underline{\Gamma}^\lambda d^4q}{(\bar{p}-\bar{q})^2} &= \oint \frac{\partial_q{}^\mu \underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} - \int_{\bar{p}\neq\bar{q}} [\partial_{q\,\mu}(\bar{p}-\bar{q})^{-2}][\partial_q{}^\mu \underline{\Gamma}^\lambda] d^4q \\
&= \oint \frac{\partial_q{}^\mu \underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} - \oint [\partial_q{}^\mu (\bar{p}-\bar{q})^{-2}] \underline{\Gamma}^\lambda d\sigma_\mu \\
&\quad + \int_{\bar{p}\neq\bar{q}} [\Box_q^2 (\bar{p}-\bar{q})^{-2}] \underline{\Gamma}^\lambda d^4q \ . \qquad (D.28)
\end{aligned}
$$

Notice that the last integral in Eq.(D.28) vanishes as can be seen by looking at Eq.(D.20) and the fact that we are excluding the point $\bar{p}=\bar{q}$ in the integration. Using Eq.(D.16) once again, and replacing $\partial_q{}^\mu$ by $-\partial^\mu$ we obtain,

$$
\int_{\bar{p}\neq\bar{q}} \frac{\Box_q^2 \underline{\Gamma}^\lambda d^4q}{(\bar{p}-\bar{q})^2} = \oint \frac{\partial_q{}^\mu \underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} + \partial_\mu \oint \frac{\underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} \ . \qquad (D.29)
$$

Subtituting this expression into Eq.(D.27) yields the result,

$$\underline{\Gamma}^{\lambda}(\bar{p}_1,\bar{p}_2) = \underline{\gamma}^{\lambda} - \frac{i}{(2\pi)^2}\oint \frac{\partial_q{}^{\mu}\underline{\Gamma}^{\lambda} d\sigma_{\mu}}{(\bar{p}-\bar{q})^2} - \frac{i}{(2\pi)^2}\partial^{\mu}\oint \frac{\underline{\Gamma}^{\lambda} d\sigma_{\mu}}{(\bar{p}-\bar{q})^2}$$
$$+ \frac{i}{(2\pi)^2}\Big\{ -\tfrac{1}{4}\epsilon\not{\nabla}\oint \ln(\bar{p}-\bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_{\nu}$$
$$- \tfrac{1}{4}\not{\nabla}\oint \ln(\bar{p}-\bar{q})^2 d\not{\sigma}\Box_q^2\underline{\Gamma}^{\lambda}$$
$$+ \tfrac{1}{8}\not{\nabla}\oint \ln(\bar{p}-\bar{q})^2 d\not{\sigma}\underline{\gamma}_{\nu}\underline{F}^{\lambda\nu}\Big\} \,. \qquad (D.30)$$

Since $\underline{S}(\bar{q}_1)$ and $\underline{S}(\bar{q}_2)$ appear in Eq.(D.30) in $\underline{F}^{\lambda\nu}$, we might expect singularities at $(\bar{q} \pm \bar{k})^2 = m^2 = 1$; that is at $\bar{q}^2 \pm 2\bar{q}\cdot\bar{k} + \bar{k}^2 = 1$. At this point it is convenient to make a Wick rotation.[31] This rotation causes the real axis of the fourth component of the momentum to lie along the imaginary axis. This can be done by making the substitution,

$$p^0 \to i\,p^4; \qquad d^4p = dp^0dp^1dp^2dp^3 \to i dp^1dp^2dp^3dp^4 \,. \qquad (D.31)$$

Making this transformation,it is possible to write the condition at which we can expect a singularity as

$$\bar{q}^2 \pm 2\bar{q}\cdot\bar{k} + \bar{k}^2 = -s \mp 2\sqrt{s(-\bar{k}^2)}\zeta - (-\bar{k}^2) = 1 \,. \qquad (D.32)$$

The quantity ζ is the cosine of the angle between the 4-vectors $\bar{q}$ and $\bar{k}$. where $-s = q_{\mu}q^{\mu} = \bar{q}^2$. If s and $(-\bar{k}^2)$ are always positive, the minimum value of the left-hand side occurs at $\pm\zeta = -1$. Thus, we conclude that

$$s \pm 2\sqrt{s(-\bar{k}^2)}\zeta \geq s - 2\sqrt{s(-\bar{k}^2)} + (-\bar{k}^2) = [\sqrt{s} - \sqrt{-\bar{k}^2}]^2 \geq 0 \,,$$

and therefore

$$-s \mp 2\sqrt{s(-\bar{k}^2)} - (-\bar{k}^2) \leq 0 < 1 \,. \qquad (D.33)$$

Thus, the Wick rotation and limitation to positive values of s and $(-\bar{k}^2)$ eliminates the singularities.

The surfaces over which we integrate in Eq.(D.30) exclude the point ($\bar{p} = \bar{q}$) by an infinitesimal sphere. Therefore, we can use an infinitesimal sphere of radius ϵ' and let this radius approach zero in the end. Since $d\sigma^\mu$ is the outward drawn normal, it points *toward* the singularity at $\bar{p} = \bar{q}$,

$$d\sigma^\mu = -\frac{(q^\mu - p^\mu)}{\epsilon'}\epsilon'^3 d\Omega \ . \tag{D.34}$$

Notice that all the surface integrals in Eq.(D.30), except the second, have the form,

$$\begin{aligned} I_1 &= \oint_{|\bar{q}-\bar{p}|=\epsilon'} f(|\bar{q}-\bar{p}|^2) g_\nu d\sigma^\nu \\ &= -g_\nu f(\epsilon')\epsilon'^2 \oint (q^\nu - p^\nu) d\Omega = 0 \ . \end{aligned} \tag{D.35}$$

The second integral, however, is

$$\begin{aligned} I_2 &= \partial^\mu \oint_{|\bar{q}-\bar{p}|=\epsilon'} \frac{\underline{\Gamma}^\lambda(\bar{q}_1,\bar{q}_2)}{(\bar{p}-\bar{q})^2} d\sigma_\mu = \epsilon'^2 \partial^\mu \oint \frac{(p_\mu - q_\mu)\underline{\Gamma}^\lambda d\Omega}{(\bar{p}-\bar{q})^2} \\ &= \epsilon'^2 \oint_{|\bar{q}-\bar{p}|=\epsilon'} \frac{2\underline{\Gamma}^\lambda(\bar{q}_1,\bar{q}_2) d\Omega}{(\bar{p}-\bar{q})^2} \\ &= 2 \oint_{|\bar{q}-\bar{p}|=\epsilon'} \underline{\Gamma}(\bar{q}_1,\bar{q}_2) d\Omega \ . \end{aligned} \tag{D.36}$$

In the limit $\bar{q} \to \bar{p}$, (*i.e.* $\epsilon' \to 0$),

$$I_2 = 2\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) \oint d\Omega = 2(2\pi^2)\underline{\Gamma}^\lambda(\bar{p}_1,\bar{p}_2) \ . \tag{D.37}$$

When these integrals are subtituted in Eq.(D.30), the vertex function $\underline{\Gamma}^\lambda$ on the left-hand side is canceled leading to the equation,

$$\begin{aligned} 0 = \underline{\gamma}^\lambda + \frac{1}{(2\pi)^2}\Bigg\{ &\oint_{\Re} \frac{\partial_q{}^\mu \underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} + \partial^\mu \oint_{\Re} \frac{\underline{\Gamma}^\lambda d\sigma_\mu}{(\bar{p}-\bar{q})^2} \\ &+ \tfrac{1}{4}\not{\nabla} \oint_{\Re} \ln(\bar{p}-\bar{q})^2 d\not{\sigma} \Box_q^2 \underline{\Gamma}^\lambda + \tfrac{1}{4}\epsilon\not{\nabla} \oint_{\Re} \ln(\bar{p}-\bar{q})^2 \underline{F}^{\lambda\nu} d\sigma_\nu \\ &- \tfrac{1}{8}\epsilon\not{\nabla} \oint_{\Re} \ln(\bar{p}-\bar{q})^2 d\not{\sigma} \underline{\gamma}_\nu \underline{F}^{\lambda\nu} \Bigg\} \ , \end{aligned} \tag{D.38}$$

where the integration is over the 4-dimensional outer surface $\Re$ that bounds the volume in q-space. Using the identity,

$$\partial^{\mu}\ln(\bar{p}-\bar{q})^2 = 2\frac{(p^{\mu}-q^{\mu})}{(\bar{p}-\bar{q})^2}, \tag{D.39}$$

it is possible to rewrite Eq.(D.38) in the form,

$$\begin{aligned}0 = \underline{\gamma}^{\lambda} + \frac{1}{(2\pi)^2}\Bigg\{ & \oint_{\Re}\frac{\partial_q{}^{\mu}\underline{\Gamma}^{\lambda}d\sigma_{\mu}}{(\bar{p}-\bar{q})^2} - 2\oint_{\Re}\frac{\underline{\Gamma}^{\lambda}(p^{\mu}-q^{\mu})d\sigma_{\mu}}{(\bar{p}-\bar{q})^4} \\ & + \tfrac{1}{2}\oint_{\Re}\frac{(\not{p}-\not{q})d\not{\sigma}\Box_q^2\underline{\Gamma}^{\lambda}}{(\bar{p}-\bar{q})^2} + \tfrac{1}{2}\epsilon\oint_{\Re}\frac{(\not{p}-\not{q})\underline{F}^{\lambda\nu}d\sigma_{\mu}}{(\bar{p}-\bar{q})^2} \\ & - \tfrac{1}{4}\epsilon\oint_{\Re}\frac{(\not{p}-\not{q})d\not{\sigma}\underline{\gamma}_{\nu}\underline{F}^{\lambda\nu}}{(\bar{p}-\bar{q})^2}\Bigg\}.\end{aligned} \tag{D.40}$$

If we take the surface $\Re$ to be a large sphere of radius $\sqrt{s}$ where

$$(\bar{p}-\bar{q})^2 = (p_{\mu}-q_{\mu})(p^{\mu}-q^{\mu}) \rightarrow q_{\mu}q^{\mu} = -s\ , \tag{D.41}$$

it follows that on the sphere then

$$d\sigma_{\nu} = s^{3/2}\frac{q_{\nu}}{\sqrt{s}} = s\,q_{\nu}d\Omega_q \tag{D.42}$$

where the symbol $d\Omega_q$ represents a differential solid angle in q-space. Substituting these results in Eq.(D.40) gives

$$\begin{aligned}0 = \underline{\gamma}^{\lambda} + \frac{1}{(2\pi)^2}\Bigg\{ & \oint_{\Re}(-s)^{-1}(\partial_q{}^{\mu}\underline{\Gamma}^{\lambda})s\,q_{\mu}d\Omega + 2\oint_{\Re}s^{-2}\underline{\Gamma}^{\lambda}q^{\mu}s\,q_{\mu}d\Omega_q \\ & + \tfrac{1}{2}\oint_{\Re}(-s)^{-1}(-\not{q})s\,\not{q}\Box_q^2\underline{\Gamma}^{\lambda}d\Omega_q + \tfrac{1}{2}\epsilon\oint_{\Re}(-s)^{-1}(-\not{q})\underline{F}^{\lambda\nu}s\,q_{\nu}d\Omega_q \\ & - \tfrac{1}{4}\epsilon\oint_{\Re}(-s)^{-1}(-\not{q})s\,\not{q}\underline{\gamma}_{\nu}\underline{F}^{\lambda\nu}d\Omega_q\Bigg\}.\end{aligned} \tag{D.43}$$

Cancelling all possible factors involving s and s^{-1} simplifies Eq.(D.43) and yields the result,

$$\begin{aligned}0 = \underline{\gamma}^{\lambda} + \frac{1}{(2\pi)^2}\Bigg\{ & -\oint_{\Re}q_{\mu}(\partial_q{}^{\mu}\underline{\Gamma}^{\lambda})d\Omega_q - 2\oint_{\Re}\underline{\Gamma}^{\lambda}d\Omega_q \\ & - \tfrac{1}{2}\oint_{\Re}s\Box_q^2\underline{\Gamma}^{\lambda}d\Omega_q + \tfrac{1}{2}\epsilon\oint_{\Re}\not{q}\,q_{\nu}\underline{F}^{\lambda\nu}d\Omega_q \\ & - \tfrac{1}{4}\epsilon\oint_{\Re}s\underline{\gamma}_{\nu}\underline{F}^{\lambda\nu}d\Omega\Bigg\}.\end{aligned} \tag{D.44}$$

Notice that on the integration surface, $\underline{\Gamma}^\lambda$ is constant, and therefore all terms involving derivatives of $\underline{\Gamma}^\lambda$ are zero. With this simplification Eq.(D.44) becomes

$$0 = \underline{\gamma}^\lambda + \frac{1}{(2\pi)^2}\Big\{ -2\oint_{\Re} \underline{\Gamma}^\lambda d\Omega_q \; + \tfrac{1}{2}\epsilon \oint_{\Re} \not{q}\, q_\nu \underline{F}^{\lambda\nu} d\Omega_q \\ - \tfrac{1}{4}\epsilon \oint_{\Re} s\underline{\gamma}_\nu \underline{F}^{\lambda\nu} d\Omega \Big\} . \tag{D.45}$$

From Eq.(D.3) we see that $\underline{F}^{\lambda\nu}$ involves $\underline{S}$ and $\underline{\Gamma}^\lambda$ where, by Eq.(2.37),

$$\underline{S}^{-1} = -A + B\not{p} \,. \tag{D.46}$$

Since the surface in the integral in Eq.(D.45) must be pushed to infinity, we can use the asymptotic form of $\underline{S}$, namely

$$\underline{S}^{-1} \sim B\not{p}; \quad \underline{S} \sim \not{p}^{-1}B^{-1} = \not{p}B^{-1}p^{-2} \,. \tag{D.47}$$

From Eq.(D.3), it follows that

$$\underline{F}^{\lambda\nu}(\bar{q}_1, \bar{q}_2) \sim \; -\frac{B^{-2}}{\bar{q}_1^2\bar{q}_2^2}\not{q}_1\underline{\Gamma}^\lambda(\bar{q}_1, \bar{q}_2)\not{q}_2\underline{\gamma}^\nu \,, \tag{D.48}$$

and therefore

$$q_\nu \underline{F}^{\lambda\nu} \sim -\frac{B^{-2}}{\bar{q}_1^2\bar{q}_2^2}\not{q}_1\underline{\Gamma}^\lambda(\bar{q}_1, \bar{q}_2)\not{q}_2\not{q} \tag{D.49}$$

and

$$\underline{\gamma}_\nu \underline{F}^{\lambda\nu} \sim -\frac{B^{-2}}{\bar{q}_1^2\bar{q}_2^2}\underline{\gamma}_\nu\not{q}_1\underline{\Gamma}^\lambda(\bar{q}_1, \bar{q}_2)\not{q}_2\underline{\gamma}^\nu \,. \tag{D.50}$$

Notice also that in the asymptotic region, $\not{q}_1 \sim \not{q}_2 \sim \not{q}$ and therefore $\bar{q}_1^2 \sim \bar{q}_2^2 \sim (-s)$. Substituting these forms into the previous equations gives,

$$q_\nu \underline{F}^{\lambda\nu} \sim (s^{-1})B^{-2}\not{q}\underline{\Gamma}^\lambda(\bar{q}, \bar{q}) \tag{D.51}$$

and

$$\underline{\gamma}_\nu \underline{F}^{\lambda\nu} \sim -(s^{-2})B^{-2}\underline{\gamma}_\nu[\not{q}\underline{\Gamma}^\lambda(\bar{q}, \bar{q})\not{q}]\underline{\gamma}^\nu \,. \tag{D.52}$$

We assume that the vertex function has the form,

$$\underline{\Gamma}^{\lambda}(\bar{q}_1,\bar{q}_2) = v\underline{\gamma}^{\lambda} + \underline{\Gamma}^{\lambda}_{A}(\bar{q}_1,\bar{q}_2) \ , \tag{D.53}$$

where $\underline{\Gamma}^{\lambda}_{A}(\bar{q}_1,\bar{q}_2) \to 0$ as $s \to \infty$. With this assumption, Eqs.(D.51) and (D.52) becomes

$$q_\nu \underline{F}^{\lambda\nu} \sim (s^{-1}) v B^{-2} \not{q} \underline{\gamma}^{\lambda} \tag{D.54}$$

and

$$\underline{\gamma}_\nu \underline{F}^{\lambda\nu} \sim -(s^{-2}) v B^{-2} \underline{\gamma}_\nu [\not{q}\underline{\gamma}^{\lambda}\not{q}]\underline{\gamma}^{\nu} \ . \tag{D.55}$$

Using the properties of the Dirac matrices (see appendix A), it is possible to transform Eq.(D.55) into the form

$$\underline{\gamma}_\nu \underline{F}^{\lambda\nu} \sim 4s^{-2} v B^{-2} q^{\lambda}\not{q} + 2s^{-1} v B^{-2} \underline{\gamma}^{\lambda} \ . \tag{D.56}$$

Substituting Eqs.(D.54) and (D.56) into Eq.(D.45) gives

$$\begin{aligned} 0 = \underline{\gamma}^{\lambda} + \frac{1}{(2\pi)^2}\Big\{ & -2(2\pi^2)\underline{\Gamma}^{\lambda} v - \tfrac{1}{2}\epsilon B^{-2}\underline{\gamma}^{\lambda} \\ & - \tfrac{1}{4}\epsilon[2B^{-2}(2\pi^2)\underline{\gamma}^{\lambda} v - \tfrac{1}{4}\epsilon s^{-1} B^{-2} \oint_{\Re} q^{\lambda}\not{q}\, d\Omega_q \Big\} \ . \end{aligned} \tag{D.57}$$

In obtaining Eq.(D.58) we have used the fact that

$$\oint_{\Re} d\Omega_q = 2\pi^2 \ . \tag{D.58}$$

The remaining integral is standard and is found to be

$$\oint_{\Re} q^{\lambda}\not{q}\, d\Omega_q = -\tfrac{1}{2}\pi^2\, s\underline{\gamma}^{\lambda} \ . \tag{D.59}$$

Subtituting this result in Eq.(D.57) and solving the equation for the function v gives

$$v \sim \left(1 + \frac{3}{8}\epsilon B^{-2}\right)^{-1} \ , \tag{D.60}$$

or since[20] $B \sim 1$, it follows that, to order $\mathcal{O}(\epsilon)$,

$$v \sim 1 - \frac{3}{8}\epsilon \ . \tag{D.61}$$

This result is in agreement with the findings of Yock.[32]

APPENDIX E

HYPERGEOMETRIC SERIES AND ASSOCIATED LEGENDRE FUNCTIONS

E-1 Hypergeometric series

If a homogeneous linear differential equation of the second order has at most three singularities, we may assume that these are at $0, \infty$, and 1. If all of these singularities are *regular*, then the equation can be reduced to the form,

$$z(1-z)\frac{d^2u}{dz^2} + [c-(a+b+1)z]\frac{du}{dz} - ab\,u = 0 \; , \qquad (E.1)$$

where a, b, and c are independent of z. This is the *hypergeometric equation.* The parameters a, b, and c are, in general, complex numbers.

Let us define

$$(a)_n = \Gamma(a+n)/\Gamma(a) \; , \qquad (E.2)$$

i.e.,

$$(a)_0 = 1, \qquad (a)_n = a(a+1)\cdots(a+n-1) \quad n = 1, 2, 3, \ldots \; . \qquad (E.3)$$

If $c \neq 0, -1, -2, \ldots$, then

$$u_1 = \sum_{n=0}^{\infty}(a)_n(b)_n\, z^n/[(c)_n\, n!] \equiv F(a, b; c; z) \qquad (E.4)$$

is a solution of Eq.$(E.1)$ which is regular at $z = 0$. If $c = -n$ where $n = 0, 1, 2, \ldots$, then

$$\begin{aligned} u_1 &= z^{n+} \sum_{m=0}^{\infty}(a+n+1)_m(b+n+1)_m\, z^m/[(n+2)_m\, m!] \\ &= z^{n+1}F(a+n+1, b+n+1; n+2; z) \qquad (E.5) \end{aligned}$$

is a solution. The function $F(a, b; c; z)$ is called the hypergeometric series of variable z with parameters a, b, and c. If $c = -m$, $(m = 0, 1, 2, \dots)$ Eq.$(E.4)$ becomes meaningless. If $a = -n$ or $b = -n$ where $n = 0, 1, 2, \dots$, and if $c = -m$ where $m = n, n+1, n+2, \dots$, then we define

$$\begin{cases} F(-n, b; -m; z) = \sum_{r=0}^{n} (-n)_r (b)_r z^r / [(-m)_r r!] \\ F(a, -n; -m; z) = \sum_{r=0}^{n} (a)_r (-n)_r z^r / [(-m)_r m!] \end{cases} . \qquad (E.6)$$

Since Eqs.$(E.5)$ and $(E.6)$ give solutions of Eq.$(E.1)$, we see that the hypergeometric equation has a solution which is a polynomial in z whenever $-a$ or $-b$ is a non-negative integer.(If $a = -m$ or $b = -m$ and $c = -n$, where $n = 0,\ 1,\ \dots$,and $m = n+1,\ n+2,\ \dots$, the series in Eq.$(E.5)$ terminates.)

If a and b are different from $0,\ -1,\ -2, \dots$, then the hypergeometric series given in Eq.$(E.4)$ (or $(E.5)$, in the case $c = -n$) converges absolutely for all values of $|z| < 1$. For $|z| = 1$ we have:

$$\begin{cases} \text{absolute convergence for } |z| = 1 \text{ if } Re(a+b-c) < 0, \\ \text{conditional convergence for } |z| = 1 \quad z \neq 1 \text{ if } 0 \leq Re(a+b-c) < 1, \\ \text{divergence if } |z| = 1 \text{ and } 1 \leq Re(a+b-c). \end{cases}$$

The n^{th}-derivative of equation (E.4) is given by the relation,

$$\frac{d^n}{dz^n} F(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; z) . \qquad (E.7)$$

The hypergeometric series satisfies the recurrence relation,

$$(c-a)(c-b) z F(a, b; c+1; z) = c\left[(2c-a-b-1)z - c + 1\right] F(a, b; c; z)$$
$$+ c(c-1)(1-z) F(a, b; c-1; z) . \qquad (E.8)$$

E-2 Associated Legendre Functions

The Legendre functions are solutions of the differential equation,

$$(1-z^2)\frac{d^2w}{dz^2} - 2z\frac{dw}{dz} + [\nu(\nu+1) - \frac{\mu^2}{1-z^2}]w = 0\ , \tag{E.9}$$

where μ, ν, and z are unrestricted.

Under the substitution, $w = (1-z^2)^{\frac{1}{2}\mu}v$, Eq.($E$.9) becomes

$$(1-z^2)\frac{d^2w}{dz^2} - 2(\mu+1)z\frac{dw}{dz} + (\nu-\mu)(\nu+\mu+1)v = 0\ . \tag{E.10}$$

With $\zeta = \frac{1}{2} - \frac{1}{2}z$ as the independent variable, this differential equation becomes

$$\zeta(1-\zeta)\frac{d^2v}{d\zeta^2} + (\mu+1)(1-2\zeta)\frac{dv}{d\zeta} + (\nu-\mu)(\nu+\mu+1)v = 0\ . \tag{E.11}$$

This is the same as Eq.(E.1) if we set $a = \mu-\nu$, $b = \mu+\nu+1$, and $c = \mu+1$. Hence it follows that the function,

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F(-\nu, \nu+1; 1-\mu; \tfrac{1}{2} - \tfrac{1}{2}z)\ , \tag{E.12}$$

is a solution of Eq.(E.10) provided $|1-z| < 2$. The solutions $P_\nu^\mu(z)$ are called the Associated Legendre Functions of the first kind. If we set $\zeta = z^2$, Eq.(E.10) becomes

$$4\zeta(1-\zeta)\frac{d^2v}{d\zeta^2} + [2-(4\zeta+6)\zeta]\frac{dv}{d\zeta} - (\mu-\nu)(\mu+\nu+1)v = 0 \tag{E.13}$$

which is also of the hypergeometric type with $a = \frac{1}{2}(\mu+\nu+1)$, $b = \frac{1}{2}(\mu-\nu)$, and $c = \frac{1}{2}$. Hence Eq.(E.1) has a solution of the form,

$$w = Q_\nu^\mu(z) = e^{i\mu\pi}2^{-\nu-1}\sqrt{\pi}\,\frac{\Gamma(\mu+\nu+1)}{\Gamma(\nu+\frac{3}{2})}z^{-\nu-\mu-1}(z^2-1)^{\frac{1}{2}\mu}$$
$$\times F(\tfrac{1}{2}\nu + \tfrac{1}{2}\mu + 1, \tfrac{1}{2}\mu + \tfrac{1}{2}\nu + \tfrac{1}{2}; \nu + \tfrac{3}{2}; z^{-2})\ . \tag{E.14}$$

The solutions $Q_\nu^\mu(z)$ are called the Associated Legendre Functions of the second kind. Legendre's differential equation remains unchanged if μ is replaced by $-\mu$, z by $-z$, and ν by $-\nu-1$. Therefore

$$P_\nu^{\pm\mu}(\pm z), \quad Q_\nu^{\pm\mu}(\pm z), \quad P_{-\nu-1}^{\pm\mu}(\pm z), \quad \text{and} \quad Q_{-\nu-1}^{\pm\mu}(\pm z) \tag{E.15}$$

are solutions of Eq.(E.1). Using various transformation formulas of the hypergeometric series[36] it is possible to write these functions in several ways. These are presented in Tables E.1 and E.2. The Wronskian of the functions $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ is given by the expression,

$$\begin{aligned} W\{P_\nu^\mu(z), Q_\nu^\mu(z)\} &= P_\nu^\mu(z)\frac{dQ_\nu^\mu(z)}{dz} - Q_\nu^\mu(z)\frac{dP_\nu^\mu(z)}{dz} \\ &= \frac{e^{i\pi\mu}2^{2\mu}\Gamma(1+\frac{1}{2}\mu+\frac{1}{2}\nu)\Gamma(\frac{1}{2}+\frac{1}{2}\mu+\frac{1}{2}\nu)}{(1-z^2)\Gamma(1+\frac{1}{2}\nu-\frac{1}{2}\mu)\Gamma(\frac{1}{2}+\frac{1}{2}\nu-\frac{1}{2}\mu)} \ . \end{aligned} \tag{E.16}$$

The Legendre functions satistify the following relations:

$$P_\nu^\mu(z) = P_{-\nu-1}^\mu(z) \ ; \tag{E.17}$$

$$P_\nu^\mu(z) = \frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\nu-\mu)}P_\nu^{-\mu}(z) + (2/\pi)e^{-i\pi\mu}\sin(\mu\pi)Q_\nu^\mu(z) \ ; \tag{E.18}$$

$$P_\nu^{-\mu}(z) = \frac{\Gamma(1-\mu+\nu)}{\Gamma(1+\mu+\mu)}\left\{P_\nu^\mu(z) - (2/\pi)e^{-i\pi\mu}\sin(\mu\pi)Q_\nu^\mu(z)\right\} \ ; \tag{E.19}$$

$$P_\nu^{-\mu}(z) = \frac{e^{-i\mu\pi}\Gamma(1-\mu+\nu)}{\pi\cos(\nu\pi)\Gamma(1+\mu+\mu)}\sin[\pi(\nu-\mu)]\left\{Q_\nu^\mu(z) - Q_{-\nu-1}^\mu(z)\right\} \ ; \tag{E.20}$$

$$Q_\nu^\mu(z)\sin[\pi(\nu+\mu)] - Q_{-\nu-1}^\nu(z)\sin[\pi(\nu-\mu)] = \pi e^{i\mu\pi}\cos(\nu\pi)P_\nu^\mu(z) \ ; \tag{E.21}$$

$$Q_\nu^\mu(z)\sin(\pi\mu) = \tfrac{1}{2}\pi e^{i\mu\pi}\left[P_\nu^\mu(z) - \frac{\Gamma(1+\mu+\nu)}{\Gamma(1+\nu-\mu)}P_\nu^{-\mu}(z)\right] \ ; \tag{E.22}$$

$$Q_{-\nu-1}^\mu(z) - Q_\nu^\mu(z) = e^{i\pi\mu}\cos(\nu\pi)\Gamma(1+\mu+\nu)\Gamma(\mu-\nu)P_\nu^{-\mu}(z) \ ; \tag{E.23}$$

$$Q_\nu^\mu(z) = \sqrt{2/\pi}\, e^{i\pi\mu}\Gamma(1+\mu+\nu)(z^2-1)^{-\frac{1}{4}}P_{-\mu-\frac{1}{2}}^{-\nu-\frac{1}{2}}[z(z^2-1)^{-\frac{1}{2}}] \ . \tag{E.24}$$

Table E-1 Expansions for $P_\nu^\mu(z)$

$$P_\nu^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} F(-\nu,\nu+1;1-\mu;\tfrac{1}{2}-\tfrac{1}{2}z) \tag{E.25}$$

$$P_\nu^\mu(z) = \frac{2^{-\nu}}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\frac{1}{2}\mu} (z+1)^\nu F(-\nu,-\nu-\mu;1-\mu;\frac{z-1}{z+1}) \tag{E.26}$$

$$P_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\frac{1}{2}\mu} z^{\mu+\nu} F(-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,1-\mu;1-z^{-2}) \tag{E.27}$$

$$P_\nu^\mu(z) = \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\frac{1}{2}\mu} F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,1-\mu;1-z^2) \tag{E.28}$$

$$\begin{aligned} P_\nu^\mu(z) = {} & \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\frac{1}{2}\mu}[z+\sqrt{z^2-1}]^{\nu-\mu} \\ & \times F(-\frac{1}{2}\nu-\tfrac{1}{2}\mu,\tfrac{1}{2}-\mu,1-2\mu;\tfrac{2\sqrt{z^2-1}}{z+\sqrt{z^2-1}}) \end{aligned} \tag{E.29}$$

$$\begin{aligned} P_\nu^\mu(z) = {} & \frac{1}{\Gamma(1-\mu)}2^\mu(z^2-1)^{-\frac{1}{2}\mu}[z-\sqrt{z^2-1}]^{\nu-\mu} \\ & \times F(-\nu-\mu,\tfrac{1}{2}-\mu,1-2\mu;-\tfrac{2\sqrt{z^2-1}}{z-\sqrt{z^2-1}}) \end{aligned} \tag{E.30}$$

$$\begin{aligned} P_\nu^\mu(z) = {} & \frac{2^{-\nu-1}}{\sqrt{\pi}}\frac{\Gamma(-\frac{1}{2}-\nu)}{\Gamma(-\mu-\nu)} z^{\mu-\nu-1}(z^2-1)^{-\frac{1}{2}\mu} \\ & \times F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,1+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\nu+\tfrac{3}{2};z^{-2}) \\ & + \frac{2^\nu}{\sqrt{\pi}}\frac{\Gamma(\frac{1}{2}+\nu)}{\Gamma(1+\nu-\mu)} z^{\mu+\nu}(z^2-1)^{-\frac{1}{2}\mu} \\ & \times F(-\tfrac{1}{2}\mu-\tfrac{1}{2}\nu,\tfrac{1}{2}-\tfrac{1}{2}\nu-\tfrac{1}{2}\mu,\tfrac{1}{2}-\nu;z^{-2})\ . \end{aligned} \tag{E.31}$$

Table E-2 Expansions for $e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z)$

$$e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z) = 2^\nu \frac{\Gamma(1+\nu)}{\Gamma(2\nu+2)}\Gamma(1+\mu+\nu)(z+1)^{\frac{1}{2}\mu-\nu-1}(z-1)^{-\frac{1}{2}\mu} \times F(1+\mu+\nu, 1+\nu; 2\nu+2; \tfrac{2}{1+z}) \qquad (E.32)$$

$$e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z) = 2^{-1-\nu}\sqrt{\pi}\frac{\Gamma(1+\mu+\nu)}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{-\frac{1}{2}-\frac{1}{2}\nu} \times F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, \tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, \nu+\tfrac{3}{2}; \tfrac{1}{1-z^2}) \qquad (E.33)$$

$$e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z) = 2^{-1-\nu}\sqrt{\pi}\frac{\Gamma(1+\mu+\nu)}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{-1-\mu-\nu}(z^2-1)^{\frac{1}{2}\mu} \times F(1+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, \tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, \nu+\tfrac{3}{2}; z^{-2}) \qquad (E.34)$$

$$e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z) = 2^\mu\sqrt{\pi}\frac{\Gamma(1+\mu+\nu)}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{\frac{1}{2}\mu}[z+\sqrt{z^2-1}]^{-1-\mu-\nu} \times F(\mu+\tfrac{1}{2}, 1+\mu+\nu, \nu+\tfrac{3}{2}; \tfrac{z-\sqrt{z^2-1}}{z+\sqrt{z^2-1}}) \qquad (E.35)$$

$$\begin{aligned} e^{-\frac{1}{2}\pi\mu}Q_\nu^\mu(z) = {} & 2^{-1+\mu}\Gamma(\mu)(z^2-1)^{-\frac{1}{2}\mu} \\ & \times F(\tfrac{1}{2}+\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, -\tfrac{1}{2}\nu-\tfrac{1}{2}\mu, 1-\mu; 1-z^2) \\ & + 2^{-1-\nu}\sqrt{\pi}\frac{\Gamma(1+\mu+\nu)}{\Gamma(\nu+\frac{3}{2})}(z^2-1)^{-\frac{1}{2}-\frac{1}{2}\nu} \\ & \times F(\tfrac{1}{2}+\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, -\tfrac{1}{2}\nu+\tfrac{1}{2}\mu, 1+\mu; 1-z^2) \end{aligned} \qquad (E.36)$$

Several values of μ and ν are of interest and are presented below.

$$P_\nu^{\frac{1}{2}}(z) = (2\pi)^{-\frac{1}{2}}(z^2-1)^{-\frac{1}{4}}\left\{[z+\sqrt{z^2-1}]^{\nu+\frac{1}{2}} + [z+\sqrt{z^2-1}]^{-\nu-\frac{1}{2}}\right\} \qquad (E.37)$$

$$Q_\nu^{\frac{1}{2}}(z) = i\,(2/\pi)^{\frac{1}{2}}(z^2-1)^{-\frac{1}{4}}[z+\sqrt{z^2-1}]^{-\nu-\frac{1}{2}} \qquad (E.38)$$

$$P_\nu^{-\frac{1}{2}}(z) = (2/\pi)^{\frac{1}{2}}\frac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1}\left\{[z+\sqrt{z^2-1}]^{\nu+\frac{1}{2}} - [z+\sqrt{z^2-1}]^{-\nu-\frac{1}{2}}\right\} \qquad (E.39)$$

$$Q_{\nu}^{\frac{1}{2}}(z) = i\,(2\pi)^{\frac{1}{2}}\frac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1}[z+\sqrt{z^2-1}\,]^{-\nu-\frac{1}{2}} \tag{E.40}$$

$$P_{\nu}^{-\nu}(z) = \frac{2^{-\nu}}{\Gamma(1+\nu)}(z^2-1)^{\frac{1}{2}\nu} \tag{E.41}$$

$$Q_{\nu}^{\frac{1}{2}}(z) = i\,(2\pi)^{\frac{1}{2}}\frac{(z^2-1)^{-\frac{1}{4}}}{2\nu+1}[z+\sqrt{z^2-1}\,]^{-\nu-\frac{1}{2}} \tag{E.42}$$

The contiguous Legendre functions satisfy the following relations:

$$P_{\nu}^{\mu+2}(z) + 2(\mu+1)z(z^2-1)^{-1/2}\,P_{\nu}^{\mu+1}(z) = (\nu-\mu)(\nu-\mu+1)P_{\nu}^{\mu}(z)\ ; \tag{E.43}$$

$$(2\nu+1)z\,P_{\nu}^{\mu}(z) = (\nu-\mu+1)P_{\nu+1}^{\mu}(z) + (\nu+\mu)P_{\nu-1}^{\mu}(z)\ ; \tag{E.44}$$

$$\begin{aligned}(\nu-\mu)(\nu-\mu+1)P_{\nu+1}^{\mu}(z) &- (\nu+\mu)(\nu+\mu+1)P_{\nu-1}^{\mu}(z)\\ &= (2\nu+1)\sqrt{z^2-1}\,P_{\nu}^{\mu+1}(z)\ ;\end{aligned} \tag{E.45}$$

$$P_{\nu-1}^{\mu}(z) - P_{\nu+1}^{\mu}(z) = -(2\nu+1)\sqrt{z^2-1}\,P_{\nu}^{\mu-1}(z)\ ; \tag{E.46}$$

$$P_{\nu-1(z)}^{\mu} - zP_{\nu}^{\mu}(z) = -(\nu-\mu+1)\sqrt{z^2-1}\,P_{\nu}^{\mu-1}(z)\ ; \tag{E.47}$$

$$zP_{\nu}^{\mu}(z) - P_{\nu+1}^{\mu}(z) = -(\nu+\mu)\sqrt{z^2-1}\,P_{\nu}^{\mu-1}(z)\ ; \tag{E.48}$$

$$(\nu-\mu)zP_{\nu}^{\mu}(z) - (\nu+\mu)P_{\nu-1}^{\nu}(z) = \sqrt{z^2-1}\,P_{\nu}^{\mu+1}(z)\ ; \tag{E.49}$$

$$(\nu-\mu+1)P_{\nu+1}^{\mu}(z) - (\nu-\mu+1)zP_{\nu}^{\mu}(z) = \sqrt{z^2-1}\,P_{\nu}^{\mu+1}(z)\ ; \tag{E.50}$$

$$\begin{aligned}(z^2-1)\frac{dP_{\nu}^{\mu}(z)}{dz} &= (\nu-\mu+1)P_{\nu+1}^{\mu}(z) - (\mu+\nu)zP_{\nu}^{\mu}(z)\ ;\\ &= \nu zP_{\nu}^{\mu}(z) - (\mu+\nu)P_{\nu-1}^{\mu}(z)\ .\end{aligned} \tag{E.51}$$

Using the expressions given in Tables E-1 and E-2 it is possible to find the behavior of the Legendre functions of the first and second kind near the singularities, $z = 1$ and $z = \infty$. The results are presented in Table E-3.

Table E-3 Behavior of $P_\nu^\mu(z)$ and $Q_\nu^\mu(z)$ at the Singularities.

Behavior at $z = 1$

$$P_\nu^\mu(z) \sim \frac{2^\mu}{\Gamma(1-\mu)}(z^2-1)^{-\frac{1}{2}\mu} \qquad \mu \neq 1,\ 2,\ 3, \ldots, \tag{E.52}$$

$$P_\nu^m(z) \sim \frac{2^{-\frac{1}{2}m}}{\Gamma(\nu-m+1)}(z-1)^{-\frac{1}{2}m} \qquad m = 0,\ 1,\ 2, \ldots, \tag{E.53}$$

$$e^{-\frac{1}{2}\pi\mu} Q_\nu^\mu(z) \sim 2^{-1+\frac{1}{2}\mu}\Gamma(\mu)(z-1)^{-\frac{1}{2}\mu} \qquad Re\,\mu > 0 \tag{E.54}$$

$$e^{-\frac{1}{2}\pi\mu} Q_\nu^\mu(z) \sim 2^{-1-\frac{1}{2}\mu}\Gamma(-\mu)\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu-\mu+1)}(z-1)^{\frac{1}{2}\mu} \qquad Re\,\mu < 0 \tag{E.55}$$

Behavior at $z = \infty$

$$P_\nu^\mu(z) \sim 2^\nu \frac{\Gamma(\nu+\frac{1}{2})}{\sqrt{\pi}\,\Gamma(1+\nu-\mu)} z^\nu \qquad \nu > -\tfrac{1}{2} \tag{E.56}$$

$$P_\nu^\mu(z) \sim 2^{-\nu-1} \frac{\Gamma(-\nu-\frac{1}{2})}{\sqrt{\pi}\,\Gamma(-\nu-\mu)} z^{-\nu-1} \qquad \nu < -\tfrac{1}{2} \tag{E.57}$$

$$Q_\nu^\mu(z) \sim e^{i\mu\pi} 2^{-\nu-1} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+\frac{3}{2})} z^{-\nu-1} \tag{E.58}$$

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Arthur A. Broyles

Arthur A. Broyles, Chairman
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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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